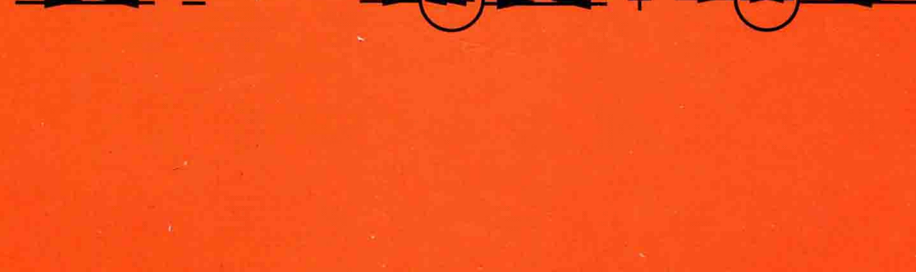
Green's Function in Condensed Matter Physics

Wang Huaiyu



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PREFACE

This book grew from a course that I have taught at Tsinghua University and Yantai University. It was originally written in Chinese and published by Science Press in Beijing. Alpha Science International Limited and Science Press have been very kind to jointly publish this English version.

The aim of this book is to integrate the elementary aspects of the Green's function theory of quantum statistics for senior students at under graduate level, graduate students and researchers who intend to enter the field of theoretical condensed matter physics. Some familiarity with quantum and statistical mechanics and solid state physics could be a prerequisite for deriving full benefit from reading this book.

In the course of writing my lecture notes and preparing the book, I frequently referred to some well-known books introducing the many-body Green's function theory. They are listed below in chronological order.

[1] Abrikosov A A, Gorkov L P, Dzaloshinski I E. Methods of Quantum Field Theory in Statistical Physics. Englewood Cliffs, New Jersey:Prentice-Hall, Inc,1962.

[2] Bonch-Bluevich V L, Tyablikov S V. The Green Function Method in Statistical Mechanics. Amsterdam:North-Holland Publishing Company,1962.

[3] Kadanoff L P, Baym C. Quantum Statistical Mechanics-Green's Function Methods in Equilibrium and Nonequilibrium Problems. New York: Addison-Wesley Publishing Company, Inc. ,1962.

[4] Fetter A L, Walecka J D. Quantum Theory of Many-Particle Systems. New York:McGraw-Hill Book Company, Inc. , 1971.

[5] Doniach S, Sondheimer E H. Green's Functions for Solid State Physicists. London:The Benjamin/Cummings Publishing Company, Inc. , 1974.

[6] Lifshitz E M, Pitaevskii L P. Statistical Physics. Vol. 9 of Course of Theoretical Physics. Oxford: Pergamon Press, 1980.

[7] Mahan G D. Many-Particle Physics. 2nd ed. New York:Plenum Press, 1990.

[8] Cai J H, Gong C D, Yao X X, et al. Green's Function Theory of Quan-

tum Statistics. Beijing: Science Press, 1982.

[9] Economou E N. Green's Functions in Quantum Physics. 3ed New York: Springer-Verlag, 2006.

[10] Wei C D, Zhang L Y, Liu F S. Green's Function Method in Solid State Physics. Beijing: Higher Education Press, 1992.

These reference books will not be specifically quoted in the text. Otherwise each may be cited for more than one time, especially Refs. [4,7-9]. I want to express my deep appreciation and gratitude here to these authors for I have extracted and used plenty of materials from these books.

This book consists of three parts. The first part, containing chapters 1 and 2, is to introduce briefly the Green's functions in mathematical physics. Although readers might be having good knowledge through the course of mathematical physics, a brief outline in the beginning of this book would be helpful for them to realize the mathematical properties of the Green's functions encountered in the subsequent two parts. The second part, containing chapters 3-7, is devoted to the one-body Green's functions. In preparing the Chinese version of this book, the first two parts greatly referred to Ref. [9], so I express here my special gratitude to the author. I found it a very good idea to incorporate the Green's functions in mathematics, one-body and many-body Green's functions into one book. This enables readers to have a comprehensive understanding of the Green's functions. This feature was hardly seen in other books in relation to the Green's function theory. Chapter 7 concerns the extension theory of the one-body Green's functions, which is considerably valuable in treating various lattice systems.

The third part, comprising remaining chapters and being the main body of this book, presents the many-body Green's function theory and some of its applications. The diagram techniques for both zero- and finite-temperature Green's functions are expressed in chapters 8-12. Chapter 13 gives three approximation schemes most often employed. The linear response theory presented in chapter 14 is extremely valid in calculating measurable quantities when a system is not far from equilibrium state. Chapter 15 introduces the equation of motion method. Chapter 16 deals with the application of the Green's function method to the Heisenberg magnetic systems. Chapters 17 and 18 take into account the systems with Bose condensation, the former for boson systems and the latter for fermion systems. The last two chapters are devoted to nonequilibrium Green's functions. Chapter 19 is the basic theory, and chapter 20 is the application in electronic transport through a mesoscopic structure, a topic that rapidly developed in recent years. With such an arrangement, it is believed that the fundamental aspects concerning the basic theory of the Green's functions have been covered.

In the book, I introduce in detail the Feynman diagram technique and the equation of motion technique which are the two mainly employed techniques in applying the many-body Green's functions. They have respective advantages. The former has clear physical significance of the approximation to any order and one is free to choose proper approximation according to the characteristics of interaction. The latter is formally much simpler than the former and has a uniform formalism for whatever system. Furthermore, the equation of motion technique is easily applicable to the nonequilibrium cases, which is its main virtue. Therefore, I give some examples of applying this technique to magnetic, superconductive and nonequilibrium systems.

Concerning the application of the many-body Green's functions, it is impossible to cover the vast areas in physics in this one volume book. I myself have worked in dealing with Heisenberg magnetic systems in recent years, and so devote much more space to it in chapter 16 than in other chapters to introduce the development in this field. The references quoted in chapters 16 and 20 are far more than in other chapters because these two chapters concern the new developments in magnetism and transport topics.

My experience in learning this course, when I was a graduate student, revealed that it was not very easy to grasp the contents. Keeping that in view, this book was written mainly for beginners, where I put down all formulas and their derivation processes in detail and in such a manner that learners of the subject are benefited to the full extent.

The exercises in this book are considered helpful for readers to understand the text. In preparing the exercises, I avoided most of those in Refs. [4,7,9], although a few were adopted. Readers are advised to do the exercises in these references in addition.

I was having great respect for George Green, an outstanding mathematician, while writing this book. In the original Chinese version, there was an Appendix E introducing his life and works. The appendix was translated and edited from English books (*Historical Note on George Green* in Ref. [5] above and Hassani S. 1999. *Mathematical Physics—A Modern Introduction to Its Foundations*. Springer-Verlag, New York, 561-562). Therefore, in this English version, it has dropped. Readers can see the two books to acquaint themselves with Green. More memorial materials with respect to Green can be found in the website http://www.nottinghamcity.gov.uk/index.aspx? articleid=1033.

I acknowledge and express my deep sense of gratitude to individuals below for their valuable discussions and help: Professors Wang Chongyu, Chen Nanxian, Yang Fumin, Xun Kun, Zhou Yunsong, Qian Meichun, Tong Dianmin, Zheng Yujun, Yu Yabin, Han Rushan, Wu Sicheng, Lin Zonghan, Zhou Bin, Guo Youjiang, Wang Zhicheng and Song Laihui.

I wish to express my thanks to my wife Miao Qing and my daughter Miao Hui for their constant understanding and support during the long terms of preparing the books in both the original and translated versions. My wife even drew part of figures for the books. I am also indebted to Profs. Miao Jichun and Wang Nianci for their detailed help in my work and life.

A special thank goes to Prof. Lu Xiukun, who taught me mathematics when I was a student in Qi Shu Yan secondary school. The friendship between us began when I stepped into the school and has lasted for over forty years.

Finally, I thank editors Hu Kai and Qian Jun for their help in publishing the original Chinese version and present English version.

The author acknowledges the support of the National Natural Science Fundation of China(grant no. 11074145).

This English version has been translated from Chinese by myself, which provides me a chance to make some corrections, clarifying changes and slight updating over the original Chinese version, where necessary. I would greatly appreciate any comments and suggestions for improvements. Although extreme care was taken to correct all the misprints, it is very likely that I have missed some (perhaps many) of them. I shall be most grateful to those readers who are kind enough to bring to my notice any remaining mistakes, typographical or otherwise for remedial action. Please feel free to contact me.

Tsinghua University, Beijing, April 2011 Wang Huaiyu
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Part I

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