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Substitutional Analysis

代换分析

[英] 卢瑟福 著

哈爾濱工業大學出版社
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内 容 简 介

本书为D·E·卢瑟福的经典著作，讲述了对称群的初步观点，主要有导数的计算、杨氏公式、基偶数导数，表的计算、正交和自然表示法，不变量二次方程、矩阵、群的特性，代换方程等内容.本书语言简洁易懂，适用于本科及研究生参考阅读.

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PREFACE

In the preparation of this book, I have been much indebted to various colleagues and friends. In particular I have received much helpful advice and constructive criticism over a period of years from Professor H. W. Turnbull, F. R. S., and Dr. W. Ledermann, all of whom have also assisted me in proof reading. I also wish to express my grateful thanks to Professor Sir Edmund Whittaker, F. R. S., and the Edinburgh University Press for their cooperation and invaluable assistance in the production of the book.

D. E. Rutherford

INTRODUCTION

The purpose of this book is to give an account of the methods employed by Alfred Young in his reduction of the symmetric group and to describe the more important results achieved by him.

The problem which first attracted Young's attention and which initiated the theory developed by him was that of solving certain substitutional equations which arose in his study of the Theory of Invariants. Although this initial problem was never far from his mind, his researches led him to study problems whose significance was deeper than he originally suspected. The keystone of these was the reduction of the symmetric group to its irreducible representations and the presentation of these representations in an explicit form. His published researches on these subjects, extending from 1900 to 1935, reveal some interesting facts. Most remarkable perhaps is the gap of twenty-five years between the second paper in 1902 and the third in 1927. In the first two papers Young had introduced the concept of a tableau which is so fundamental in the subsequent theory and had achieved some interesting results. It seems fairly certain that this brilliant inspiration was arrived at by a close study of the Gorder-Capelli series in the Theory of Invariants. This is borne out by the fact that the first use he made of his newly forged tool was its application to the Gordan-Capelli series.

In the introduction to his third paper Young writes: "When writing the two former papers I suffered from the disadvantage of being unacquainted with the closely related researches of the late Prof. Frobenius, published in the Berlin *Sitzungsberichte*, and beginning with 'Ueber Gruppencharaktere', 1896; a lucid and more elementary exposition of the main features of Frobenius's theory of group characters was given by Schur". It is uncertain at what date Young's attention was drawn to the work of Frobenius and Schur, but it is certain that their papers made a great impression on him and spurred him on to develop his own approach to the subject. To enable him to assimilate the papers of Frobenius and Schur he undertook a study of the German language. When one remembers in addition that Young was not a professional mathematician but a country clergyman with numerous clerical duties, the gap of twenty-five years between his second and third papers is not so surprising.

Adopting the point of view of abstract algebra, Thrall derives Young's orthogonal representation directly and thereby eliminates a great mass of elaborate detail which Young found necessary in constructing the orthogonal representation from the natural one.

The discovers of Frobenius and Schur are not without significance in the theory under consideration. Indeed their results embody many of those of Young. The two theories may be regarded as parallel attacks on the same problem, but in this book the emphasis will be laid on the calculus of tableaux as applied to the symmetric group, and this particular aspect of the subject is peculiar to Young's work and to that of von Neumann, Robinson and Thrall.

Likewise there have been several successful attempts, notably those of Weyl, Murnaghan and D. E. Littlewood, to relate Young's work to other branches of Modern Algebra. These, however, have already been expounded by their several authors and will not be enlarged upon here.

In this book it was considered desirable to expound the subject in terms of Young's mathematical language because in this way the theory can be studied by a reader possessing no previous knowledge of the subject apart from those portions of the Theory of Groups and the Theory of Matrices which are familiar to all mathematiclans, and because in this way only can the individuality and genius of Young be properly recognised. Nevertheless, the reader who is already familiar with Young's writings will observe that the presentation here given varies considerably from that of Young. In some cases the order of development has been changed, and in consequence of this, many of the proofs given are new. It is hoped that these changes will contribute to the lucidity and beauty of the underlying theory. It will therefore be understood that the references in the text to Young's work do not necessarily imply that the proof given is due to Young. While this is so in some cases, in others the reference is quoted only to show that the result in question was also obtained in whole or in part by Young. The references quoted in the text are given in an abridged form. The works cited are given in full in the Bibliography on page 109.

CONTENTS

THE CALCULUS OF PERMUTATIONS

Chapter I

Argument. The $n!$ possible permutations of n letters occupy a place of basic importance throughout this book. In this first chapter we shall describe some of their more interesting properties and shall introduce certain notations which will facilitate our investigations in later chapters. Most readers will find that much of this chapter is already familiar to them but they should nevertheless pay particular attention to the notations employed, especially in § 6 and § 7.

§ 1 Permutations

It is well known that there are $n!$ different permutations which can be made on n letters. We shall call these letters $z_1, \cdots, z_n$, but in most cases it will be more convenient to denote them by their suffixes only. To avoid ambiguity we shall use a special fount of type for this purpose and write $z_1, \cdots, z_n$ simply as $1, \cdots, n$. Thus

$$\begin{pmatrix} 1, 2, \cdots, n \\ i_1, i_2, \cdots, i_n \end{pmatrix} \tag{1.1}$$

where $i_1, \cdots, i_n$ denote the letters $1, \cdots, n$ in some order, denotes the permutation which changes the letter 1 into the letter i_1 and so on. There is no particular reason why the columns of this permutation should be written in any special order. The same permutation might be denoted by

$$\begin{pmatrix} 2, 1, \cdots, n \\ i_2, i_1, \cdots, i_n \end{pmatrix}$$

or, more generally, by

$$\begin{pmatrix} k_1, k_2, \cdots, k_n \\ i_{k_1}, i_{k_2}, \cdots, i_{k_n} \end{pmatrix}$$

where $k_1, \cdots, k_n$ are the letters $1, \cdots, n$ in some order. Greek letters $\sigma, \tau, \cdots$ will frequently be employed to denote permutations in cases where the above more precise notation is unnecessary. In particular we shall write

$$\varepsilon \equiv \begin{pmatrix} 1,2,\cdots,n \\ 1,2,\cdots,n \end{pmatrix}$$

throughout this book, where ε denotes the identical permutation.

In Group Theory these permutations are considered as elements which have an independent existence, but for many of our purposes they must be thought of as operators which are applicable to functions of the n letters. Thus, when the permutation (1.1) is applied to a function $F(z_1, \cdots, z_n)$, a function $F(z_{i_1}, \cdots, z_{i_n})$ is obtained which is in general different from $F(z_1, \cdots, z_n)$. This is expressed by the formula

$$\begin{pmatrix} 1,2,\cdots,n \\ i_1, i_2, \cdots, i_n \end{pmatrix} F(z_1, \cdots, z_n) = F(z_1, \cdots, z_n) \tag{1.2}$$

For this reason we shall write the result of operation on F first with the permutation σ_1 and then with the permutation σ_2 as $\sigma_2\sigma_1 F$ and not as $\sigma_1\sigma_2 F$. It is easily proved that the effect of operating successively with two permutations is equivalent to operating with a single permutation, which we may call the product of the other two. In other words, $\sigma_2\sigma_1$ is itself a permutation σ_3, namely, that permutation which results from first performing the permutation σ_1, and then performing σ_2. It should be borne in mind that in Group Theory, where the permutations are not thought of as operators acting on functions, the product which we now denote by $\sigma_2\sigma_1$ is more usually written $\sigma_1\sigma_2$. This distinction is an important one since in general permutations are non-commutative with respect to multiplication. The truth of this is evident from the following illustration

$$\begin{pmatrix} 1,2,3 \\ 2,1,3 \end{pmatrix}\begin{pmatrix} 1,2,3 \\ 1,3,2 \end{pmatrix} = \begin{pmatrix} 1,2,3 \\ 2,3,1 \end{pmatrix}$$

$$\begin{pmatrix} 1,2,3 \\ 1,3,2 \end{pmatrix}\begin{pmatrix} 1,2,3 \\ 2,1,3 \end{pmatrix} = \begin{pmatrix} 1,2,3 \\ 3,1,2 \end{pmatrix}$$

We shall see in a moment that every permutation σ possesses an inverse σ^{-1}. Since two permutations σ_1 and σ_2 do not necessarily commute, we must be careful to write

$$(\sigma_1\sigma_2)^{-1} = \sigma_2^{-1}\sigma_1^{-1}$$

as is usual in all non-commutative algebra.

§ 2 The Symmetric Group $\mathscr{S}_n$

From time to time it will be necessary to quote certain standard results from the more elementary parts of the theory of finite groups. Such results are very well known and appear in all the usual text-books. This being so, it would seem to be superfluous to include the proofs of these results in this book. An exception will nevertheless be made in the case of theorems and formulae which are specifically concerned with the $n!$ permutations of the symmetric group. In such cases there will be no objection to expressing the proofs concerned in a concise form.

Theorem 1 *The $n!$ permutations of n letters form a group.*

Proof (i) The product of any two permutations is a permutation.

(ii) The identity permutaiton

$$\varepsilon \equiv \begin{pmatrix} 1,2,\cdots,n \\ 1,2,\cdots,n \end{pmatrix}$$

which leaves every letter unaltered is the unit element of the group.

(iii) Since

$$\begin{pmatrix} i_1,\cdots,i_n \\ 1,\cdots,n \end{pmatrix}\begin{pmatrix} 1,\cdots,n \\ i_1,\cdots,i_n \end{pmatrix} = \begin{pmatrix} 1,\cdots,n \\ 1,\cdots,n \end{pmatrix} = \varepsilon$$

every permutation $\begin{pmatrix} 1,\cdots,n \\ i_1,\cdots,i_n \end{pmatrix}$

has an inverse $\begin{pmatrix} i_1,\cdots,i_n \\ 1,\cdots,n \end{pmatrix}$

(iv) It can readily be verified that the associative law holds.

The $n!$ permutations of n letters therefore satisfy the four group postulates and consequently form a group. This group which is called the *symmetric group* of order $n!$ is usually denoted by $\mathscr{S}_n$.

Theorem 2 *If σ and τ be any two permutations the product $\sigma\tau\sigma^{-1}$ is that permutation which is obtained by operating*[①] *on τ with σ.*

Proof Let

$$\tau = \begin{pmatrix} 1,\cdots,n \\ j_1,\cdots,j_n \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1,\cdots,n \\ i_1,\cdots,i_n \end{pmatrix} = \begin{pmatrix} j_1,\cdots,j_n \\ k_1,\cdots k_n \end{pmatrix}$$

---

① Operating in the sense of (1,2).

Then

$$\sigma^{-1} = \begin{pmatrix} i_1, \cdots, i_n \\ 1, \cdots, n \end{pmatrix}$$

and

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} j_1, \cdots, j_n \\ k_1, \cdots, k_n \end{pmatrix} \begin{pmatrix} 1, \cdots, n \\ j_1, \cdots, j_n \end{pmatrix} \begin{pmatrix} i_1, \cdots, i_n \\ 1, \cdots, n \end{pmatrix} = \begin{pmatrix} i_1, \cdots, i_n \\ k_1, \cdots, k_n \end{pmatrix}$$

Clearly this last permutation can be obtained from τ by operating on it with σ.

The significance of this theorem, which is of fundamental importance in our subsequent work, can perhaps best be understood by comparing the relations

$$\begin{pmatrix} 1,2,3 \\ 2,1,3 \end{pmatrix} \begin{pmatrix} 1,2,3 \\ 2,3,1 \end{pmatrix} \begin{pmatrix} 2,1,3 \\ 1,2,3 \end{pmatrix} = \begin{pmatrix} 2,1,3 \\ 1,3,2 \end{pmatrix}$$

$$\begin{pmatrix} 1,2,3 \\ 2,1,3 \end{pmatrix} F(z_1, z_2, z_3) = F(z_2, z_1, z_3)$$

§ 3 Cycles and Transpositions

The most fruitful way of investigating permutations is by expressing them as products of cycles. By a cycle is meant a permutation of the type

$$\begin{pmatrix} i_1, i_2, \cdots, i_{r-1}, i_r, i_{r+1}, \cdots, i_n \\ i_2, i_3, \cdots, i_r, i_1, i_{r+1}, \cdots, i_n \end{pmatrix}$$

which leaves the letters $i_{r+1}, \cdots, i_n$ unaltered but which permutes the letters $i_1, \cdots, i_r$ cyclically. The number r of letters permuted cyclically is called the order of the cycle. Such a cycle may be economically written

$$(i_1, \cdots, i_r)$$

It might also be written $(i_2, \cdots, i_r, i_1)$, or again $(i_r, i_1, \cdots, i_{r-1})$, or in fact in r different ways in all.

The notation just introduced has the advantage that only those letters which are affected by the cycle are displayed. It is therefore clear at a glance which letters are unaffected. Thus the cycle $(2,1,4)$ leaves the letter 3 unaltered. A set of cycles no two of which affect the same letter are said to be *independent*. It is an elementary fact of considerable importance that independent cycles commute with one another. This must be so since in any product of independent cycles no letter is affected more than once.

It follows from our definition that the rth power of a cycle of order r is always the unit element; thus if $\sigma = (i_1, \cdots, i_r)$, then $\sigma^r = \varepsilon$ and $\sigma^{r-1} = \sigma^{-1}$. The inverse of a cycle of order r is a cycle of order r, namely that which permutes the letters $i_1, \cdots, i_r$ cyclically in the reverse order. These properties will be clarified by the following illustration

$$\sigma = (1,2,\cdots,r) = \begin{pmatrix} 1,2,\cdots,r,r+1,\cdots,n \\ 2,3,\cdots,1,r+1,\cdots,n \end{pmatrix}$$

$$\sigma^2 = (1,2,\cdots,r)^2 = \begin{pmatrix} 1,2,\cdots,r,r+1,\cdots,n \\ 3,4,\cdots,2,r+1,\cdots,n \end{pmatrix}$$

$$\vdots$$

$$\sigma^{r-1} = (1,2,\cdots,r)^{r-1} = \begin{pmatrix} 1,2,\cdots,r,r+1,\cdots,n \\ r,1,\cdots,r-1,r+1,\cdots,n \end{pmatrix} = (r,r-1,\cdots,1)$$

$$\sigma^r = (1,2,\cdots,r)^r = \begin{pmatrix} 1,2,\cdots,r,r+1,\cdots,n \\ 1,2,\cdots,r,r+1,\cdots,n \end{pmatrix} = \varepsilon$$

Every permutation can be expressed as a product of independent cycles. The method of doing so is typified by the following example

$$\begin{pmatrix} 1,2,3,4,5,6,7 \\ 3,6,2,7,5,1,4 \end{pmatrix} = \begin{pmatrix} 1,3,2,6,4,7,5 \\ 3,2,6,1,7,4,5 \end{pmatrix}$$

$$= \begin{pmatrix} 1,3,2,6,4,7,5 \\ 3,2,6,1,4,7,5 \end{pmatrix} \begin{pmatrix} 4,7,1,3,2,6,5 \\ 7,4,1,3,2,6,5 \end{pmatrix} \begin{pmatrix} 5,1,3,2,6,4,7 \\ 5,1,3,2,6,4,7 \end{pmatrix}$$

$$= (1,3,2,6)(4,7)(5)$$

Since any cycle of order unity is merely the identity permutation ε we may omit such cycles from any product. The permutation illustrated above is most concisely written $(1,3,2,6)(4,7)$.

Using this notation the $3!$ permutations of $\mathscr{S}_3$ are

$$\begin{pmatrix} 1,2,3 \\ 1,2,3 \end{pmatrix} = \varepsilon, \quad \begin{pmatrix} 1,2,3 \\ 2,3,1 \end{pmatrix} = (1,2,3), \quad \begin{pmatrix} 1,2,3 \\ 3,1,2 \end{pmatrix} = (3,2,1)$$

$$\begin{pmatrix} 1,2,3 \\ 1,3,2 \end{pmatrix} = (2,3), \quad \begin{pmatrix} 1,2,3 \\ 3,2,1 \end{pmatrix} = (3,1), \quad \begin{pmatrix} 1,2,3 \\ 2,1,3 \end{pmatrix} = (1,2)$$

The group Table 1 which tabulates all products $\sigma\tau$ of this group is given below. The elements σ are in the column on the left and the elements τ lie in the row at the top.

Table 1

$\mathscr{S}_3$	ε	$(1,2,3)$	$(3,2,1)$	$(2,3)$	$(3,1)$	$(1,2)$
ε	ε	$(1,2,3)$	$(3,2,1)$	$(2,3)$	$(3,1)$	$(1,2)$
$(1,2,3)^{-1}=(3,2,1)$	$(3,2,1)$	ε	$(1,2,3)$	$(3,1)$	$(1,2)$	$(2,3)$
$(3,2,1)^{-1}=(1,2,3)$	$(1,2,3)$	$(3,2,1)$	ε	$(1,2)$	$(2,3)$	$(3,1)$
$(2,3)^{-1}=(2,3)$	$(2,3)$	$(3,1)$	$(1,2)$	ε	$(1,2,3)$	$(3,2,1)$
$(3,1)^{-1}=(3,1)$	$(3,1)$	$(1,2)$	$(2,3)$	$(3,2,1)$	ε	$(1,2,3)$
$(1,2)^{-1}=(1,2)$	$(1,2)$	$(2,3)$	$(3,1)$	$(1,2,3)$	$(3,2,1)$	ε

A cycle of order two is called a *transposition* and is evidently its own inverse. It can be readily verified that

$$\begin{aligned}(1,2,\cdots,r) &= (1,r)(1,r-1)\cdots(1,3)(1,2)\\ &= (1,2)(2,3)\cdots(r-2)(r-1)(r-1,r)\end{aligned}$$

Many other such resolutions are possible; e. g.

$$(1,2,\cdots,r) = (2,3,\cdots,r,1) = (2,1)(2,r)\cdots(2,3)$$

but it must be remembered that transpositions like cycles do not commute if they have letters in common. It follows that any permutation, being a product of cycles, can always be expressed as a product of transpositions. Thus

$$\begin{pmatrix}1,2,3,4,5,6,7\\3,6,2,7,5,1,4\end{pmatrix} = (1,3,2,6)(4,7) = (1,3)(3,2)(2,6)(4,7)$$

We are now in a position to prove the following result.

Theorem 3 *Every permutation can be expressed as a product of transpositions of the form* $(k-1,\ k)$ *where* $k-1$ *and* k *are two consecutive letters.*

Proof Since we have already expressed any permutation as a product of transpositions, it remains to show that any transposition (i,j) can be expressed as a product of transpositions of the form $(k-1,\ k)$. From theorem 2(§2) it follows that

$$(i,j) = (1,i)(1,j)(1,i)$$

and that

$$(1,i) = (i,\cdots,3,2)(1,2)(2,3,\cdots,i)$$

Also, we have seen that

$$\begin{aligned}(i,\cdots,3,2) &= (i,i-1)\cdots(4,3)(3,2)\\ (2,3,\cdots,i) &= (2,3)(3,4)\cdots(i-1,i)\end{aligned}$$

A combination of these formulae yields the desired result.

For example

$$(2,4) = (1,2)(1,4)(1,2) = (1,2)(3,4)(2,3)(1,2)(2,3)(3,4)(1,2)$$

§ 4 Odd and Even Permutations

When any permutation σ is applied to the alternant

$$\Delta \equiv \prod_{j>i} (z_j - z_i)$$

the value of this expression remains unaltered apart from sign. Now for any given σ we must have either $\sigma\Delta = +\Delta$ or else $\sigma\Delta = -\Delta$. If $\sigma\Delta = +\Delta$ we call σ an even permutation, whereas if $\sigma\Delta = -\Delta$ we call σ an odd permutation. Clearly every transposition is an odd permutation, for when it is applied to Δ it changes the sign of an odd number of factors. It follows from this that the product of an even number of transpositions is an even permutation and that the product of an odd number of transpositions is an odd permutation. Although in general any permutation can be expressed in a variety of ways as the product of transpositions, it is clear from the foregoing that each such way involves an even number of transpositions if the permutation be an even one, but an odd number of transpositions if the permutation be odd. In illustration we remark that ε, $(1,2,3)$, $(1,2)(3,4)$ are even permutations and that $(1,2)$, $(1,2,3,4)$ are odd permutations.

Associated with each permutation σ we now define a number $\mathscr{S}_\sigma$ with the properties

$$\zeta_\sigma = +1, \textit{if } \sigma \textit{ is an even permutation}$$
$$\zeta_\sigma = -1, \textit{if } \sigma \textit{ is an odd permutation}$$

This notation will prove very useful in subsequent chapters. It is already familiar in the theory of determinants; e. g.

$$\begin{vmatrix} x_1 \cdots x_n \\ \vdots \\ t_1 \cdots t_n \end{vmatrix} = \sum_\sigma \sigma(x_1, \cdots, t_n)$$

§ 5 Classes of Permutations

The concept of a class of elements is a very important one in the Theory of Groups and we shall make some use of it at a later stage in this book. For our purposes it will suffice to consider the case of the symmetric group $\mathscr{S}_n$ only. Two permu-

tations τ_1 and τ_2 of $\mathscr{S}_n$ are said to belong to the same class if it is possible to find a permutation σ of $\mathscr{S}_n$ such than

$$\sigma\tau_1\sigma^{-1} = \tau_2$$

Theorem 2(§ 2) tells us that this is possible if and only if τ_2 can be obtained by operating on τ_1 with some permutation σ. This means that τ_1 and τ_2 must be built up of the same number of independent cycles and that the orders of these component cycles are the same in each case although the arrangement of the letters in the cycles will be different when τ_1 and τ_2 are distinct. In other words, all those permutations which are the products of independent cycles of orders $\alpha_1, \cdots, \alpha_k$ form a class of $\mathscr{S}_n$ and no other permutations of $\mathscr{S}_n$ belong to this class. In particular, $\mathscr{S}_3$ has three classes, namely

$$\varepsilon;\ (1,2,3),(3,2,1);\ (2,3),(3,1),(1,2)$$

Since we have shown that the inverse of a cycle of order r is a cycle of order r, it is patent that σ and σ^{-1} must always belong to the same class of $\mathscr{S}_n$. A corresponding statement is not true for every finite group.

If τ_1 is expressible as the product of r transpositions, then $\sigma\tau_1\sigma^{-1}$ is also expressible as the product of r transpositions. It follows from this that all the permutations of a given class are either all odd permutations or else are all even permutations.

§ 6 Substitutional Expressions

The substitutional expressions which we now introduce can be viewed from two angles. Although the permutations σ, τ considered as elements of the group $\mathscr{S}_n$ admit of only one law of combination, namely multiplication, yielding products such as $\sigma\tau$ and $\tau\sigma$, we can attach a meaning to $\sigma + r$ if we regard the permutations as operators acting on a function F of the n letters $z_1, \cdots, z_n$. If $\sigma F \equiv F_\sigma$, $\tau F \equiv F_\tau$, we define $\sigma + \tau$ to be that operation which yields the function $F_\sigma + F_\tau$ when it is applied to the function F. In view of this definition we can extend the above notation and write

$$F_\sigma + F_\tau \equiv F_{\sigma+\tau}$$

We can generalise the foregoing idea by attaching numerical coefficients to the permutations. The general *substitutional expression* has the form

$$X = \lambda_1\varepsilon + \lambda_2\sigma_2 + \cdots + \lambda_{n!}\sigma_{n!}$$

where $\varepsilon, \sigma_2, \cdots, \sigma_{n!}$ are the $n!$ distinct permutations of $\mathscr{S}_n$ and where $\lambda_1, \cdots, \lambda_{n!}$ are numerical coefficients. X is defined as that operation which when applied to any function F yields F_X, where

$$F_X \equiv \lambda_1 F + \lambda_2 F_{\sigma_2} + \cdots + \lambda_{n!} F_{\sigma_n}$$

and where $F_{\sigma_i} \equiv \sigma_i F$. These substitutional expressions form the main topic of this book.

To take a more abstract point of view we may define the substitutional expressions X as hyper-complex numbers constructed from the permutations σ_i of $\mathscr{S}_n$ as units. The resulting algebra is sometimes called the *group algebra* of $\mathscr{S}_n$. Any two substitutional expressions $X \equiv \sum_i \lambda_{\sigma_i}\sigma_i$ and $Y \equiv \sum_i \mu_{\sigma_i}\sigma_i$ have a sum

$$X + Y \equiv \sum_i (\lambda_{\sigma_i} + \mu_{\sigma_i})\sigma_i$$

and a product

$$XY \equiv \sum_{i,j} \lambda_{\sigma_i}\mu_{\sigma_j}\sigma_i\sigma_j$$

If we write $\sigma_i\sigma_j = \sigma_k$, then $\sigma_j = \sigma_i^{-1}\sigma_k$; thus we may also write

$$XY \equiv \sum_{i,j} \lambda_{\sigma_i}\mu_{\sigma_i^{-1}\sigma_k}\sigma_k$$

so that not only the sum $X + Y$ but also the product XY is a linear combination of the permutations σ_i. As such it is clear that the sum and the product of two substitutional expressions are also substitutional expressions. As illustrations of this the reader may verify, with the help of the group table for $\mathscr{S}_3$(§ 3), that if

$$X = \varepsilon - 2(1,2), \quad Y = 3(1,2,3) + (1,2)$$

then

$$X + Y = \varepsilon - (1,2) + 3(1,2,3)$$
$$XY = -2\varepsilon + (1,2) - 6(2,3) + 3(1,2,3)$$
$$YX = -2\varepsilon + (1,2) - 6(3,1) + 3(1,2,3)$$

§ 7 The Positive and Negative Symmetric Groups on r Letters

Certain types of substitutional expressions are of frequent occurrence and as such deserve a special notation. The set of all permutations σ obtainable by permuting the letters $i_1, \cdots, i_r$ among themselves constitute a symmetric group of order $r!$. To be more precise, we shall call it the *positive symmetric group* on the r letters $i_1, \cdots, i_r$. The sum of all the elements of this positive symmetric group will be denoted by

$$\{i_1, \cdots, i_r\}$$

and we can denote the positive symmetric group itself by

$$\mathscr{G}\{i_1,\cdots,i_r\}$$

It is easily proved that the substitutional expressions $\zeta_\sigma\sigma$, where σ ranges over the permutations of $G\{i_1,\cdots,i_r\}$, also constitute a group which is simply isomorphic with $\mathscr{G}\{i_1,\cdots,i_r\}$. This group is called the negative symmetric group on the r letters $\{i_1,\cdots,i_r\}$ and will be denoted by

$$\mathscr{G}\{i_1,\cdots,i_r\}'$$

The sum of its elements can be written

$$\{i_1,\cdots,i_r\}'$$

From the abstract point of view $\mathscr{G}\{i_1,\cdots,i_r\}$ and $\mathscr{G}\{i_1,\cdots,i_r\}'$ are merely different representations of $\mathscr{S}_r$. The elements of these representations are substitutional expressions and in the general case $\{i_1,\cdots,i_r\}$ and $\{i_1,\cdots,i_r\}'$ are distinct elements of the group algebra. For example

$$\{1,2,3\}=\varepsilon+(1,2,3)+(3,2,1)+(2,3)+(3,1)+(1,2)$$
$$\{1,2,3\}'=\varepsilon+(1,2,3)+(3,2,1)-(2,3)-(3,1)-(1,2)$$

It is a fundamental fact in the Theory of Groups that when each element of the group is pre-or post-multiplied by some specified element of the group, the resulting products comprise all the elements of the group, each occurring once only. It fellows from this that if π be any element of $\mathscr{G}\{i_1,\cdots,i_r\}$ and $\nu=(\zeta_\pi\pi)$ be any element of $\mathscr{G}\{i_1,\cdots,i_r\}'$, then

$$\pi\{i_1,\cdots,i_r\}=\{i_1,\cdots,i_r\}\pi=\{i_1,\cdots,i_r\}$$
$$\nu\{i_1,\cdots,i_r\}'=\{i_1,\cdots,i_r\}'\nu=\{i_1,\cdots,i_r\}'$$

Here it is to be observed that if the letters $i_1,\cdots,i_r$ all belong to the set $1,\cdots,n$, the π is necessarily and element of $\mathscr{S}_n$. So is ν if it be an even element, but if it is odd then it is $-\nu$ that is an element of $\mathscr{S}_n$.

If $\mathscr{H}$ is a sub-group of order m of the group $\mathscr{G}\{i_1,\cdots,i_r\}$, then it is known from the Theory of Groups that elements $\pi_2,\cdots,\pi_{r!/m}$ of $\mathscr{G}\{i_1,\cdots,i_r\}$ can be found such that the elements of

$$\mathscr{H},\pi_i\mathscr{H},\cdots,\pi_{r!/m}\mathscr{H}$$

include all those of $\mathscr{G}\{i_1,\cdots,i_r\}$ once each and no others. Now if i_h, i_k be two letters of the set $i_1,\cdots,i_r$, then $\mathscr{G}\{i_h,\cdots,i_k\}$ is a sub-frow of $\mathscr{G}\{i_1,\cdots,i_r\}$. If follows that elements $\pi_2,\cdots,\pi_{r!/2}$ can be found such that

$$\{i_1,\cdots,i_r\}=(\varepsilon+\pi_2+\cdots+\pi_{r!/2})\{i_h,\cdots,i_k\}$$

This factor $\{i_h,i_k\}$ is equal to $\varepsilon+(i_h,i_k)$. It may be written either on the left or the right of the other factor $(\varepsilon+\pi_2+\cdots+\pi_{r!/2})$ as desired. A very similar argument would show that $\{i_1,i_r\}'$ has a factor $\{i_h,i_k\}'$ or $\varepsilon-(i_h,i_k)$. This factor can also