

**ALM 13**

**Advanced Lectures in Mathematics**

# **Handbook of Geometric Analysis (Vol. II)**

几何分析手册 (第II卷)

Editors: Lizhen Ji • Peter Li • Richard Schoen • Leon Simon



高等教育出版社  
HIGHER EDUCATION PRESS

**ALM 13**

**Advanced Lectures in Mathematics**

# **Handbook of Geometric Analysis (Vol. II)**

几何分析手册 (第 II 卷)

Jihe Fenxi Shouce (Di II Juan)

Editors: Lizhen Ji • Peter Li • Richard Schoen • Léon Simon



高等教育出版社 • 北京  
HIGHER EDUCATION PRESS BEIJING



International Press

## 图书在版编目 (CIP) 数据

几何分析手册. 第2卷 = Handbook of Geometric Analysis (Vol II): 英文/(美) 季理真等编. —北京: 高等教育出版社, 2010.4

ISBN 978-7-04-028883-4

I. ①几… II. ①季… III. ①几何-数学分析-手册-英文 IV. ①O18-62

中国版本图书馆 CIP 数据核字 (2010) 第 021224 号

Copyright © 2010 by

**Higher Education Press**

4 Dewai Dajie, Beijing 100120, P. R. China, and

**International Press**

387 Somerville Ave, Somerville, MA, U.S.A

*All rights reserved. No part of this book may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without permission.*

策划编辑 王丽萍

责任编辑 王丽萍

封面设计 张申申

版式设计 范晓红

责任校对 王 雨

责任印制 毛斯璐

出版发行 高等教育出版社  
社 址 北京市西城区德外大街 4 号  
邮政编码 100120  
总 机 010-58581000

经 销 蓝色畅想图书发行有限公司  
印 刷 北京中科印刷有限公司

购书热线 010-58581118  
免费咨询 400-810-0598  
网 址 <http://www.hep.edu.cn>  
<http://www.hep.com.cn>  
网上订购 <http://www.landaco.com>  
<http://www.landaco.com.cn>  
畅想教育 <http://www.widedu.com>

开 本 787×1092 1/16  
印 张 28.5  
字 数 690 000

版 次 2010 年 4 月第 1 版  
印 次 2010 年 4 月第 1 次印刷  
定 价 78.00 元

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换。

版权所有 侵权必究

物料号 28883-00

# ADVANCED LECTURES IN MATHEMATICS

## EXECUTIVE EDITORS

Shing-Tung Yau  
Harvard University  
Cambridge, MA. USA

Lizhen Ji  
University of Michigan  
Ann Arbor, MI. USA

Kefeng Liu  
University of California, Los Angeles  
Los Angeles, CA. USA  
Zhejiang University  
Hangzhou, China

## EXECUTIVE BOARD

Chongqing Cheng  
Nanjing University  
Nanjing, China

Tatsien Li  
Fudan University  
Shanghai, China

Zhong-Ci Shi  
Institute of Computational Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Zhiying Wen  
Tsinghua University  
Beijing, China

Zhouping Xin  
The Chinese University of Hong Kong  
Hong Kong, China

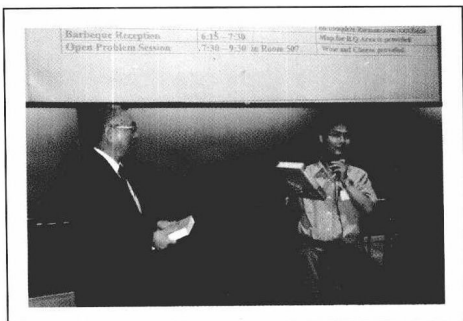
Lo Yang  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Weiping Zhang  
Nankai University  
Tianjin, China

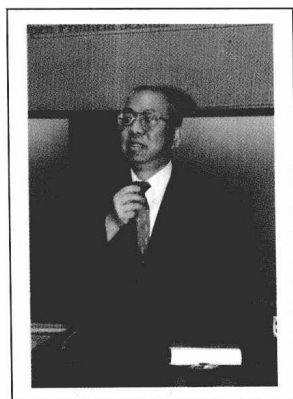
Xiangyu Zhou  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Xiping Zhu  
Sun Yat-sen University  
Guangzhou, China

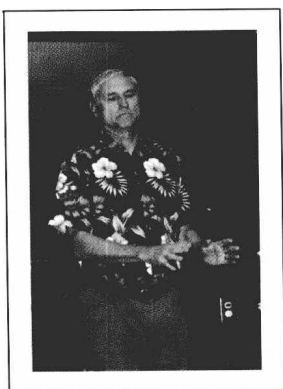
**Dedicated to Shing-Tung Yau on the occasion of  
his sixtieth birthday.**



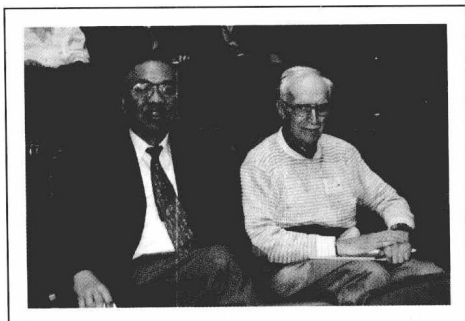
Lizhen Ji handed a copy of *Handbook of Geometric Analysis, Vol. I* to Prof. Yau



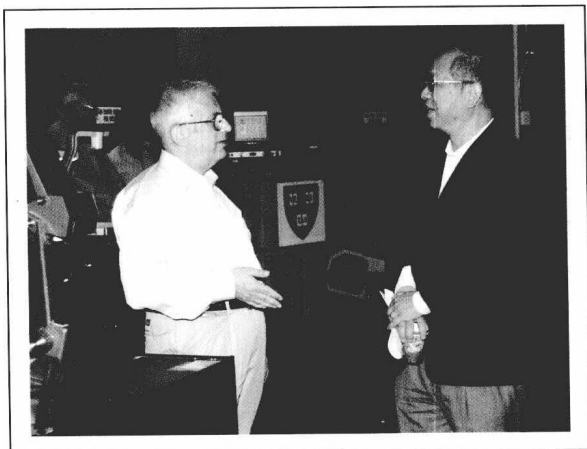
Shing-Tung Yau



Richard Schoen

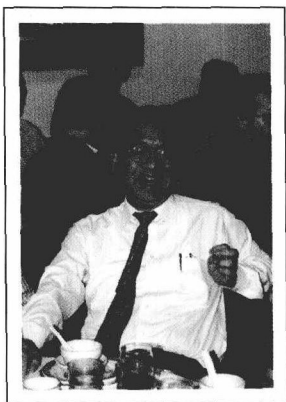


Shing-Tung Yau and Eugenio Calabi



Grigory Margulis and Shing-Tung Yau

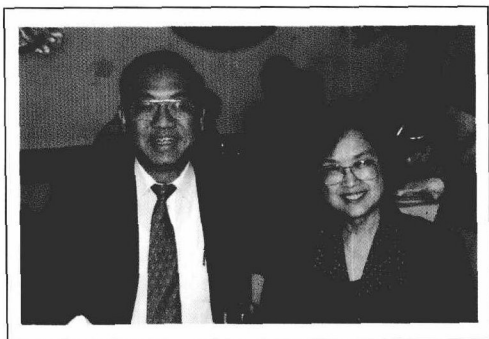
All pictures were taken during the conference "Geometric Analysis: Present and Future" Aug27, Sept 1, 2008, most of the pictures were taken by Jeff Mozzochi and Conan Leung.



Shing-Tung Yau



Karen Uhlenbeck, Gerhard Huisken, Richard Hamilton and Shing-Tung Yau



Shing-Tung Yau and his wife



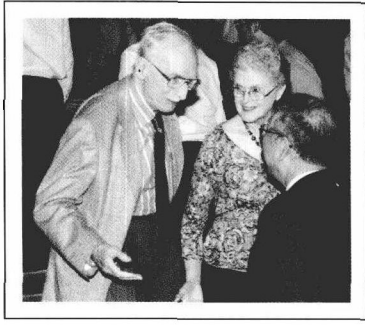
Peter Li and his wife



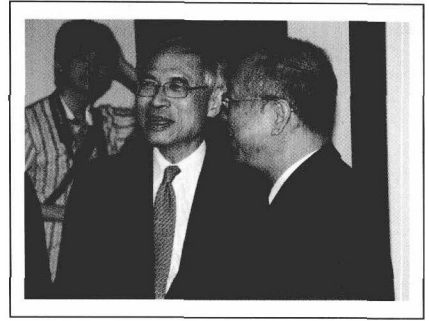
Shing-Tung Yau and his wife, Richard Schoen and his wife, and Leon Simon



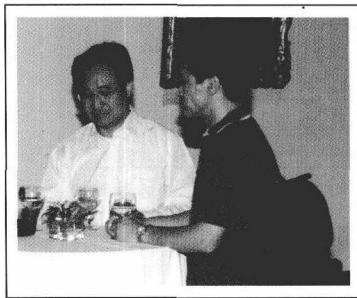
Shing-Tung Yau and Tristan Hubsch



Mr. and Mrs. Calabi and Shing-Tung Yau



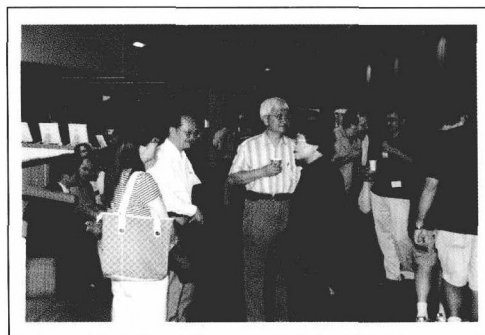
Lawrence J. Lau and Shing-Tung Yau



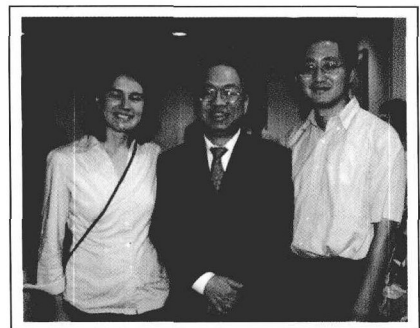
Xiping Zhu and Shouwu Zhang



Tianjun Li and Conan Leung

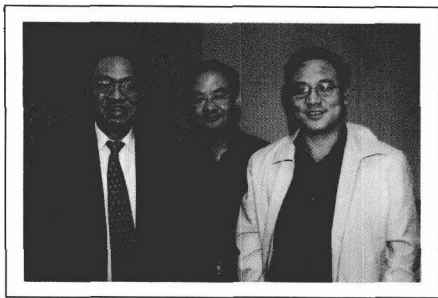


Mingzhang Kang, Jin Yu, Chulian Terng and others  
in the Hall

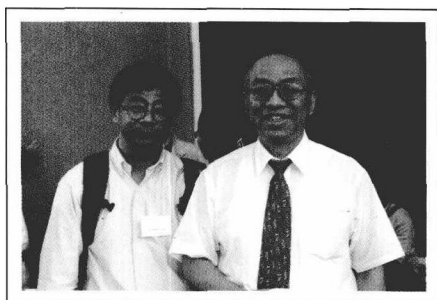


Alina Marian, Shing-Tung Yau and Jun Li





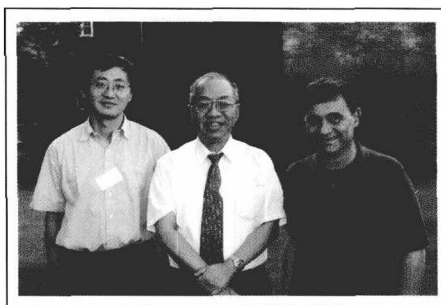
Shing-Tung Yau, Dexin Kong and Kefeng Liu



Huai-dong Cao and Shing-Tung Yau



Shing-Tung Yau and Lixin Qin



Jun Li, Shing-Tung Yau and Andrey Todorov

## Preface

The marriage of geometry and analysis, in particular non-linear differential equations, has been very fruitful. An early deep application of geometric analysis is the celebrated solution by Shing-Tung Yau of the Calabi conjecture in 1976. In fact, Yau together with many of his collaborators developed important techniques in geometric analysis in order to solve the Calabi conjecture. Besides solving many open problems in algebraic geometry such as the Severi conjecture, the characterization of complex projective varieties, and characterization of certain Shimura varieties, the Calabi-Yau manifolds also provide the basic building blocks in the superstring theory model of the universe. Geometric analysis has also been crucial in solving many outstanding problems in low dimensional topology, for example, the Smith conjecture, and the positive mass conjecture in general relativity.

Geometric analysis has been intensively studied and highly developed since 1970s, and it is becoming an indispensable tool for understanding many parts of mathematics. Its success also brings with it the difficulty for the uninitiated to appreciate its breadth and depth. In order to introduce both beginners and non-experts to this fascinating subject, we have decided to edit this handbook of geometric analysis. Each article is written by a leading expert in the field and will serve as both an introduction to and a survey of the topics under discussion. The handbook of geometric analysis is divided into several parts, and this volume is the second part.

Shing-Tung Yau has been crucial to many stages of the development of geometric analysis. Indeed, his work has played an important role in bringing the well-deserved global recognition by the whole mathematical sciences community to the field of geometric analysis. In view of this, we would like to dedicate this handbook of geometric analysis to Shing-Tung Yau on the occasion of his sixtieth birthday.

Summarizing the main mathematical contributions of Yau will take many pages and is probably beyond the capability of the editors. Instead, we quote several award citations on the work of Yau.

The citation of the Veblen Prize for Yau in 1981 says: "*We have rarely had the opportunity to witness the spectacle of the work of one mathematician affecting, in a short span of years, the direction of whole areas of research.... Few mathematicians can match Yau's achievements in depth, in impact, and in the diversity of methods and applications.*"

In 1983, when Yau was awarded a Fields medal, L. Nirenberg described Yau's work up to that point:

*"Yau has done extremely deep work in global geometry and elliptic partial differential equations, including applications in three-dimensional topology and in general relativity theory. He is an analyst's geometer (or geometer's analyst) with remarkable technical power and insight. He has succeeded in solving problems on which progress had been stopped for years."*

More than ten years later, Yau was awarded the Carfoord prize in 1994, and the citation of the award says:

*"The Prize is awarded to ... Shing-Tung Yau, Harvard University, Cambridge, MA, USA, for his development of non-linear techniques in differential geometry leading to the solution of several outstanding problems."*

*Thanks to Shing-Tung Yau's work over the past twenty years, the role and understanding of the basic partial differential equations in geometry has changed and expanded enormously within the field of mathematics. His work has had an impact on areas of mathematics and physics as diverse as topology, algebraic geometry, representation theory, and general relativity as well as differential geometry and partial differential equations. Yau is a student of legendary Chinese mathematician Shiing-Shen Chern, for whom he studied at Berkeley. As a teacher he is very generous with his ideas and he has had many students and also collaborated with many mathematicians."*

In 2010, Yau was awarded the Wolf Prize for his lifetime achievements in geometric analysis and mathematical physics, and the award citation probably gives one of the best summaries of his major works up to 2010:

*"Shing-Tung Yau (born 1949, China) has linked partial differential equations, geometry, and mathematical physics in a fundamentally new way, decisively shaping the field of geometric analysis. He has developed new analytical tools to solve several difficult nonlinear partial differential equations, particularly those of the Monge-Ampere type, critical to progress in Riemannian, Kahler and algebraic geometry and in algebraic topology, that radically transformed these fields. The Calabi-Yau manifolds, as these are known, a particular class of Kahler manifolds, have become a cornerstone of string theory aimed at understanding how the action of physical forces in a high-dimensional space might ultimately lead to our four-dimensional world of space and time. Prof. Yau's work on T-duality is an important ingredient for mirror symmetry, a fundamental problem at the interface of string theory and algebraic and symplectic geometry. While settling the positive mass and energy conjectures in general relativity, he also created powerful analytical tools, which have broad applications in the investigation of the global geometry of space-time."*

*Prof. Yau's eigenvalue and heat kernel estimates on Riemannian manifolds count among the most profound achievements of analysis on manifolds. He studied minimal surfaces, solving several classical problems, and then used his results, to create a novel approach to geometric topology. Prof. Yau has been exceptionally productive over several decades, with results radiating onto many areas of pure and applied*

*mathematics and theoretical physics.*

*In addition to his diverse and fundamental mathematical achievements, which have inspired generations of mathematicians, Prof. Yau has also had an enormous impact, worldwide, on mathematical research, through training an extraordinary number of graduate students and establishing several active mathematical research centers."*

Indeed, he has already trained more than 60 Ph.D. students.

We wish Yau a happy sixtieth birthday and continuing success in many years to come!

Lizhen Ji  
Peter Li  
Richard Schoen  
Leon Simon

# Contents

## Heat Kernels on Metric Measure Spaces with Regular Volume Growth

<i>Alexander Grigor'yan</i> .....	1
1 Introduction .....	1
1.1 Heat kernel in $\mathbb{R}^n$ .....	2
1.2 Heat kernels on Riemannian manifolds .....	3
1.3 Heat kernels of fractional powers of Laplacian .....	4
1.4 Heat kernels on fractal spaces .....	5
1.5 Summary of examples .....	7
2 Abstract heat kernels .....	8
2.1 Basic definitions .....	8
2.2 The Dirichlet form .....	11
2.3 Identifying $\Phi$ in the non-local case .....	13
2.4 Volume of balls .....	17
3 Besov spaces .....	21
3.1 Besov spaces in $\mathbb{R}^n$ .....	21
3.2 Besov spaces in a metric measure space .....	23
3.3 Embedding of Besov spaces into Hölder spaces .....	24
4 The energy domain .....	26
4.1 A local case .....	26
4.2 Non-local case .....	31
4.3 Subordinated heat kernel .....	32
4.4 Bessel potential spaces .....	35
5 The walk dimension .....	36
5.1 Intrinsic characterization of the walk dimension .....	36
5.2 Inequalities for the walk dimension .....	39
6 Two-sided estimates in the local case .....	46
6.1 The Dirichlet form in subsets .....	46
6.2 Maximum principles .....	47
6.3 A tail estimate .....	47
6.4 Identifying $\Phi$ in the local case .....	55
References .....	57

**A Convexity Theorem and Reduced Delzant Spaces**

<i>Bong H. Lian, Bailin Song</i> .....	<b>61</b>
1 Introduction .....	61
2 Convexity of image of moment map .....	64
3 Rationality of moment polytope .....	69
4 Realizing reduced Delzant spaces .....	74
5 Classification of reduced Delzant spaces .....	82
References .....	94

**Localization and some Recent Applications**

<i>Bong H. Lian, Kefeng Liu</i> .....	<b>97</b>
1 Introduction .....	97
2 Localization .....	100
3 Mirror principle .....	102
4 Hori-Vafa formula .....	112
5 The Mariño-Vafa Conjecture .....	115
6 Two partition formula .....	123
7 Theory of topological vertex .....	125
8 Gopakumar-Vafa conjecture and indices of elliptic operators .....	128
9 Two proofs of the ELSV formula .....	129
10 A localization proof of the Witten conjecture .....	132
11 Final remarks .....	134
References .....	134

**Gromov-Witten Invariants of Toric Calabi-Yau Threefolds**

<i>Chiu-Chu Melissa Liu</i> .....	<b>139</b>
1 Gromov-Witten invariants of Calabi-Yau 3-folds .....	139
1.1 Symplectic and algebraic Gromov-Witten invariants .....	139
1.2 Moduli space of stable maps .....	139
1.3 Gromov-Witten invariants of compact Calabi-Yau 3-folds .....	140
1.4 Gromov-Witten invariants of noncompact Calabi-Yau 3-folds .....	141
2 Traditional algorithm in the toric case .....	142
2.1 Localization .....	142
2.2 Hodge integrals .....	143
3 Physical theory of the topological vertex .....	144
4 Mathematical theory of the topological vertex .....	146
4.1 Locally planar trivalent graph .....	146
4.2 Formal toric Calabi-Yau (FTCY) graphs .....	148
4.3 Degeneration formula .....	150
4.4 Topological vertex .....	152
4.5 Localization .....	153
4.6 Framing dependence .....	154
4.7 Combinatorial expression .....	154

4.8 Applications .....	155
4.9 Comparison .....	155
5 GW/DT correspondences and the topological vertex .....	156
Acknowledgments .....	156
References .....	156

## Survey on Affine Spheres

<i>John Loftin</i> .....	161
1 Introduction .....	161
2 Affine structure equations .....	163
3 Examples .....	164
4 Two-dimensional affine spheres and Titeica's equation .....	165
5 Monge-Ampère equations and duality .....	168
6 Global classification of affine spheres .....	172
7 Hyperbolic affine spheres and invariants of convex cones .....	173
8 Projective manifolds .....	176
9 Affine manifolds .....	181
10 Affine maximal hypersurfaces .....	185
11 Affine normal flow .....	186
References .....	187

## Convergence and Collapsing Theorems in Riemannian Geometry

<i>Xiaochun Rong</i> .....	193
Introduction .....	193
1 Gromov-Hausdorff distance in space of metric spaces .....	194
1.1 The Gromov-Hausdorff distance .....	194
1.2 Examples .....	199
1.3 An alternative formulation of GH-distance .....	202
1.4 Compact subsets of $(\mathcal{M}et, d_{GH})$ .....	204
1.5 Equivariant GH-convergence .....	206
1.6 Pointed GH-convergence .....	209
2 Smooth limits-fibrations .....	217
2.1 The fibration theorem .....	217
2.2 Sectional curvature comparison .....	219
2.3 Embedding via distance functions .....	223
2.4 Fibrations .....	226
2.5 Proof of theorem 2.1.1 .....	231
2.6 Center of mass .....	234
2.7 Equivariant fibrations .....	235
2.8 Applications of the fibration theorem .....	240
3 Convergence theorems .....	245
3.1 Cheeger-Gromov's convergence theorem .....	245
3.2 Injectivity radius estimate .....	248
3.3 Some elliptic estimates .....	253
3.4 Harmonic radius estimate .....	255

3.5	Smoothing metrics .....	259
4	Singular limits-singular fibrations .....	260
4.1	Singular fibrations .....	261
4.2	Controlled homotopy structure by geometry .....	265
4.3	The $\pi_2$ -finiteness theorem .....	269
4.4	Collapsed manifolds with pinched positive sectional curvature .....	271
5	Almost flat manifolds .....	273
5.1	Gromov's theorem on almost flat manifolds .....	273
5.2	The Margulis lemma .....	275
5.3	Flat connections with small torsion .....	277
5.4	Flat connection with a parallel torsion .....	281
5.5	Proofs—part I .....	285
5.6	Proofs—part II .....	290
5.7	Refined fibration theorem .....	294
	References .....	297
 <b>Geometric Transformations and Soliton Equations</b>		
	<i>Chuu-Lian Terng</i> .....	<b>301</b>
1	Introduction .....	301
2	The moving frame method for submanifolds .....	306
3	Line congruences and Bäcklund transforms .....	309
4	Sphere congruences and Ribaucour transforms .....	315
5	Combescure transforms, O-surfaces, and $k$ -tuples .....	317
6	From moving frame to Lax pair .....	320
7	Soliton hierarchies constructed from symmetric spaces .....	329
8	The $\frac{U}{K}$ -system and the Gauss-Codazzi equations .....	336
9	Loop group actions .....	343
10	Action of simple elements and geometric transforms .....	347
	References .....	355
 <b>Affine Integral Geometry from a Differentiable Viewpoint</b>		
	<i>Deane Yang</i> .....	<b>359</b>
1	Introduction .....	359
2	Basic definitions and notation .....	361
2.1	Linear group actions .....	361
3	Objects of study .....	362
3.1	Geometric setting .....	362
3.2	Convex body .....	362
3.3	The space of all convex bodies .....	362
3.4	Valuations .....	362
4	Overall strategy .....	363
5	Fundamental constructions .....	363
5.1	The support function .....	363
5.2	The Minkowski sum .....	364



5.3	The polar body .....	365
5.4	The inverse Gauss map .....	366
5.5	The second fundamental form .....	366
5.6	The Legendre transform .....	366
5.7	The curvature function .....	367
6	The homogeneous contour integral .....	368
6.1	Homogeneous functions and differential forms .....	368
6.2	The homogeneous contour integral for a differential form ....	369
6.3	The homogeneous contour integral for a measure .....	369
6.4	Homogeneous integral calculus .....	373
7	An explicit construction of valuations .....	374
7.1	Duality .....	375
7.2	Volume .....	375
8	Classification of valuations .....	376
9	Scalar valuations .....	376
9.1	$SL(n)$ -invariant valuations .....	376
9.2	Hug's theorem .....	378
10	Continuous $GL(n)$ -homogeneous valuations .....	378
10.1	Scalar valuations .....	378
10.2	Vector-valued valuations .....	379
11	Matrix-valued valuations .....	380
11.1	The Cramer-Rao inequality .....	381
12	Homogeneous function- and convex body-valued valuations .....	383
13	Questions .....	384
	References .....	385

## Classification of Fake Projective Planes

<i>Sai-Kee Yeung</i> .....	<b>391</b>
1 Introduction .....	391
2 Uniformization of fake projective planes .....	393
3 Geometric estimates on the number of fake projective planes ....	396
4 Arithmeticity of lattices associated to fake projective planes .....	398
5 Covolume formula of Prasad .....	410
6 Formulation of proof .....	411
7 Statements of the results .....	419
8 Further studies .....	423
References .....	427