Mathematics Monograph Series 29

Complex Differences and Difference Equations

Chen Zongxuan（陈宗煊）

（复域差分与差分方程）

SCIENCE PRESS
Beijing

中国科学院科学出版基金资助出版

Mathematics Monograph Series 29

Complex Differences and Difference Equations

(复域差分与差分方程)

Chen Zongxuan (陈宗煊)

SCIENCE PRESS
Beijing

Responsible Editor: Li Xin

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-041875-3 (Beijing)

Preface

As early as 30 years ago, several initial results on the existence of meromorphic solutions of some complex difference equations have been obtained by Bank S. B., Kaufman R. P., Shimomura S. and Yanagihara N., et al.. Later on, researches in this field were developed slowly, almost in a state of stagnation. Until in recent 10 years, Nevanlinnas value distribution theory has been used as a powerful tool to study complex differences and difference equations, which results in the rapid development in this field. Thus, complex differences and difference equations has attracted more and more attentions, and becomes a hot and new direction. The essential reason is the penetration and application of the value distribution theory.

In fact differences and difference equations developing from real field to complex field is an inevitable result. The expansion to complex number field can help us obtain more essential cognition. However so far, the complex differences and difference equations are still in the early stages and further research is needed. Therefore, it is very necessary to encourage research in this field and promote its development.

Furthermore, as we know many mathematical models in physics, chemistry, quantum mechanics and economics are basically differential equations and difference equations. In complex number field, there are little results on the complex difference equations, contrasted sharply with the rich results on complex differential equations. Complex difference equations becomes popular just in recent years. So, it is necessary to strengthen this research. Many regular patterns in natural sciences and economics are formed with difference equations. For example, the model of growth of population is actually a difference equation: the Pielou logistic equation. The function $\Gamma(z)$ widely used in the physics and mechanics satisfies a linear difference equation, and some new properties of $\Gamma(z)$ can be obtained by the study of linear difference equations.

The author of this book is one of the earliest domestic researchers engaged in this study, and has been supported by the National Natural Science Foundation of China many times. He brought up a lot of doctor and master graduate students with this research subject, and uses the new achievements of this research at home and abroad as the main content of the seminars. The manuscript is based on the content in our seminars, which was gradually formed, improved and perfected. It contains the basic theory and new results on this research field in recent years, as well as the author and his students research results.

So far there hasn't been any book focusing on this topic at home or a broad. Meanwhile, many complex analysis experts all over the world are engaged in this research, as well as more graduate students. The aim of writing this book is to provide an induction or guidance for readers, to help them understand and master the basic theory and research methods, to make readers interested in this field read new documents as soon as possible so that they can study some new problems and get the solutions of unsolved questions.

Any reader who is familiar with the value distribution theory can read this book successfully. Chapter 1 mainly introduces the basic properties of complex differences, including the difference analogue of the logarithmic derivative; the estimates on the integral counting function and characteristic function of the shifts of a meromorphic function; and the difference analogue of Nevanlinnas second fundamental theorem, Clunie Lemma and Mohonko Lemma. Chapters 2, 3, 4 mainly provide the relationships between meromorphic functions and their differences, shifts and higher differences; the zeros and fixed points of difference and divided differences of meromorphic functions with small growth; properties of difference polynomials. Chapters 5 to 9 are devoted to the analytic properties of linear and non-linear difference equations in the complex plane. Chapter 10 deals with the value distribution of q-differences and the analytic properties of meromorphic solutions of linear and non-linear q-difference equations. Chapter 11 is concerned with the uniqueness problems of meromorphic functions concerning their shifts and difference operators.

This is the first book about complex differences and difference equations both at home and abroad. If there are any defects and mistakes, please give us understanding and we welcome any suggestions and criticisms.

Author would like to express his thanks to his students Huang Zhibo, Xiao Lipeng, Mao Zhiqiang, Zheng Xiumin, Zhang Ranran, Chen Baoqin, Peng Changwen, Jiang Yeyang, Lan Shuangtin, Chen Chuangxin, Cui Ning, Wang Jun, Liu Huifang, Li Sheng, Tu Jin for their cooperations and contributions to the study of complex analysis during the joint working period. Author would like to express his thanks to professor Kwang Ho Shon for his cooperations and contributions to the study of complex analysis during the joint working period.

Author would like to express his thanks to professors Wang Yuefei and Hasi Wulan for their recommendation to publish this book in Science Press by the Foundation of Science Press (No. 2014-003).

This book was supported by the Foundation of Science Press (No. 2014-003) and the National Natural Science Foundation of China (No. 11171119).

Chen Zongxuan
May 11th, 2013

List of Symbols

$\mathbb{C}$	The set of all finite complex numbers.
$\widehat{\mathbb{C}}$	$\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$.
Re z	The real part of a complex number z.
Im z	The imaginary part of a complex number z.
$\Delta f(z)$	The difference operator $\Delta f(z) = f(z+1) - f(z)$.
$\Delta_c f(z)$	The difference operator $\Delta_c f(z) = f(z+c) - f(z)$ where $c \in \mathbb{C} \backslash \{0\}$.
$\Delta^n f(z)$	The forward differences $\Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z)$, $n = 1, \ \ldots$.
$M(r, \ f)$	$M(r, \ f) = \max\{\lvert f(z)\rvert : \ \lvert z\rvert = r\}$.
$m(r, f)$	The proximity function of a meromorphic function $f(z)$ $$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \lvert f(re^{i\theta})\rvert d\theta,$$ where $\log^+ x = \max\{0, \log x\}$ for all $x > 0$.
$n(r, f)$	The number of poles of f, including multiplicities, for $\lvert z\rvert \leqslant r$.
$N(r, \ f)$	$N(rf) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r$.
$T(r, \ f)$	$T(r,f) = m(r,f) + N(r,f)$.
$\sigma(f)$	The order of growth of $f(z)$.
$\mu(f)$	The lower order of growth of $f(z)$.
$\sigma_2(f)$	The hyper order of $f(z)$, is defined as $$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$
$\lambda(f(z))$	The exponent of convergence of zeros of $f(z)$.
$\lambda\left(\frac{1}{f(z)}\right)$	The exponent of convergence of poles of $f(z)$.
$\tau(f(z))$	The exponent of convergence of fixed points of $f(z)$ is defined as $$\tau(f(z)) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f(z)-z}\right)}{\log r}.$$
$S(r, \ f)$	$S(r, \ f) = O(\log T(r, \ f) + \log r)$ for all r outside of a possible exceptional set with finite logarithmic measure.
$S(f)$	The set of functions, such that if $g \in S(f)$, then $T(r, \ g) = S(r, \ f)$, that is, if $g \in S(f)$, then g is called small compared to f, or slowly moving with respect to f.
$\widehat{S(f)}$	$\widehat{S}(f) = S(f) \bigcup \{\infty\}$.

$\mu(r)$	The maximum term of f, if $f(z)=\sum_{n=0}^{\infty} a_n z^n$, then $\mu(r)=\mu(r,\ f):=\max\{\|a_n\|r^n:\ n\geqslant 0\}$.
$\nu(r)$	The central index of f which is defined as $\nu(r)=\nu(r,\ f)=\max\{n:\|a_n\|r^n=\mu(r,\ f)$ for all $n\geqslant 0\}$.
$m(E)$	The linear measure of a set $E\subset(0,\infty), m(E)=\int_E dt$.
$lm(E)$	The logarithmic measure of a set $E\subset(1,\infty)$, $lm(E)=\int_E \frac{dt}{t}$.

Contents

Chapter 1

Basic Properties of Complex Differences

1.1 Preliminaries

The Nevanlinna characteristic $T(r,\ f)$, which encodes information about the distribution of values of f on the disk $|z| \leqslant r$, plays a central role in the theory of meromorphc functions. It is a sum of two parts:

$$T(r,\ f) = m(r,\ f) + N(r,\ f).$$

The proximity function $m(r,\ f)$ is given by

$$m(r,\ f) = m(r,\ \infty,\ f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{0,\ \log x\}$. The proximity function is an averaged measure of how large f becomes on the circle $|z| = r$. Define the counting function $n(r,\ f)$ to be the number of poles of (counted according to multiplicities) in the circle $|z| = r$. The integrated counting function, $N(r,\ f)$, is then defined to be

$$N(r,\ f) = N(r,\ \infty,\ f) = \int_0^r \frac{n(t,\ f) - n(0,\ f)}{t} dt + n(0,\ f)\log r.$$

The Nevanlinna characteristic function $T(r,\ f)$ is therefore the sum of a measure of how large f becomes on $|z| = r$ and a measure of the number of poles of f in $|z| < r$.

Now, we recall the following results which play important roles in investigation of complex differences and difference equations.

Theorem 1.1.1(The first main theorem of Nevanlinna theory, see [89, 185]) *Let f be a meromorphic function, $a \in \mathbb{C}$. Then*

$$T(r,\ f) = T\left(r,\ \frac{1}{f-a}\right) + O(1).$$

The first main theorem implies that if $f(z)$ takes a value $a \in \mathbb{C}\bigcup\{\infty\}$ fewer times than average, i.e., the counting function $N(r,\ a) = N\left(r,\ \frac{1}{f-a}\right)$ is relatively small, then the proximity function $m(r,\ a) = m\left(r,\ \dfrac{1}{f-a}\right)$ must be large, and vice versa. Loosely speaking, if a meromorphic function assumes a certain value a relatively few times, the values of $f(z)$ are "near" the value a in a large part of the complex plane.

Theorem 1.1.2(The second main theorem of Nevanlinna theory, see [89, 185]) *Let f be a meromorphic function, $a_j \in \mathbb{C}\bigcup\{\infty\}$ $(j = 1, \ldots, q,\ q \geqslant 3)$ be each other not equal. Then*

$$(q-2)T(r,\ f) \leqslant \sum_{j=1}^{q} N\left(r,\ \frac{1}{f-a_j}\right) + S(r,\ f),$$

where $S(r,\ f) = O(\log T(r,\ f) + \log r)$.

Theorem 1.1.3(The logarithmic derivative lemma, see [89, 185]) *Let f be a meromorphic function. Then*

$$m\left(r,\ \frac{f'(z)}{f(z)}\right) = S(r,\ f)$$

holds outside a possible set of finite linear measure.

It shows that the proximity function of logarithmic derivative of $f(z)$ grows more slowly than the characteristic function of $f(z)$.

Remark 1.1.1 A meromorphic function f is transcendental if and only if

$$\lim_{r\to\infty} \frac{T(r,\ f)}{\log r} = \infty.$$

Let $f(z)$, $g(z)$ and $h(z)$ be meromorphic functions, and $n \in \mathbb{N}$. Then

$$\begin{aligned}
&T(r,\ f+g) \leqslant T(r,\ f) + T(r,\ g) + O(1),\\
&T(r,\ fg) \leqslant T(r,\ f) + T(r,\ g),\\
&T(r,\ f^n) = nT(r,\ f),\\
&T(r,\ fg+gh+hf) \leqslant T(r,\ f) + T(r,\ g) + T(r,\ h) + O(1),\\
&T(r+1,\ f) = T(r,\ f) + S(r,\ f) \text{ if } f \text{ is of finite order.}
\end{aligned}$$

Theorem 1.1.4(The Milloux theorem, see [89, 185]) *Let f be a meromorphic function. Then*

$$T(r,\ f) \leqslant \overline{N}(r,\ f) + N\left(r,\ \frac{1}{f}\right) + N\left(r,\ \frac{1}{f'-1}\right) + S(r,\ f).$$

Theorem 1.1.5(The Wiman-Valiron theory, see [96, 121]) *Let f be a transcen-*

dental entire function, let $0 < \delta < \dfrac{1}{4}$ *and* z *be such that* $|z| = r$ *and that*

$$|f(z)| > M(r,\ f)\nu(r,\ f)^{-\frac{1}{4}+\delta}$$

holds. Then there exists a set $F \subset \mathbb{R}_+$ *of finite logarithmic measure. i.e.,* $\displaystyle\int_F \frac{dt}{t} < +\infty$, *such that*

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{\nu(r,\ f)}{z}\right)^m (1 + o(1))$$

holds for all $m \geqslant 0$ *and all* $r \notin F$.

Remark 1.1.2 Let f be an entire function whose Taylor expansion is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The maximum term is defined as

$$\mu(r) = \mu(r,\ f) := \max\{|a_n|r^n :\ n \geqslant 0\};$$

the central index is defined as

$$\nu(r) = \nu(r,\ f) = \max\{n : |a_n|r^n = \mu(r,\ f)\ \text{ for all } n \geqslant 0\}.$$

Theorem 1.1.6(Lemma of Clunie, see [89, 121]) *Let* $f(z)$ *be a transcendental meromorphic solution of equation*

$$f^n P(z,\ f) = Q(z,\ f),$$

where $P(z,\ f)$ *and* $Q(z,\ f)$ *are polynomials in* f *and its derivatives with meromorphic coefficients, say* $\{a_\lambda :\ \lambda \in I\}$, *such that* $m(r,\ a_\lambda) = S(r,\ f)$ *for all* $\lambda \in I$. *If the total degree of* $Q(z,\ f)$ *as a polynomial in* f *and its derivatives is* $\leqslant n$, *then*

$$m(r,\ P(z,\ f)) = S(r,\ f).$$

Theorem 1.1.7(Mohon'ko, see [136, 121]) *Let* $f(z)$ *be a meromorphic funtion. Then for all irreducible rational funtions in* f,

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\displaystyle\sum_{i=0}^{p} a_i(z) f^i}{\displaystyle\sum_{j=0}^{q} b_j(z) f^j},$$

such that the meromorphic coefficients $a_i(z), b_j(z)$ *satisfies*

$$\begin{cases} T(r,\ a_i) = S(r,\ f), & i = 0, 1, \ldots, p, \\ T(r,\ b_j) = S(r,\ f), & j = 0, 1, \ldots, q, \end{cases}$$

then we have

$$T(r,\ R(z,f)) = \max\{p,q\}\cdot T(r,\ f) + S(r,\ f).$$

Theorem 1.1.8(The Weierstrass factorization theorem, see [121]) *Let $f(z)$ be an entire function, with a zero of multiplicity $m \geqslant 0$ at $z=0$. Let the other zeros of f be $a_1,\ a_2,\ \ldots,$ each zero being repeated as many times as its multiplicity implies. Then f has the representation*

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_{m_n}\left(\frac{z}{a_n}\right)$$

for some entire function g and some integers m_n. If $(a_n)_{n\in\mathbb{N}}$ has a finite exponent of convergence λ, then m_n may be taken as $k=[\lambda]>\lambda-1$.

Remark 1.1.3 The Weierstrass primary factors is defined as

$$\begin{cases} E_0(z) := 1-z, \\ E_m(z) := (1-z)\exp\left(z+\dfrac{z^2}{2}+\ldots+\dfrac{z^m}{m}\right), \quad m\in\mathbb{N}. \end{cases}$$

If $Q(z) = z^m \prod_{n=1}^{\infty} E_{m_n}\left(\dfrac{z}{a_n}\right)$, then $\lambda(Q)=\sigma(Q)=\lambda(f)$.

Theorem 1.1.9(See [121]) *Let $g:\ (0,\ +\infty)\to\mathbb{R},\ h:\ (0,\ +\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ outside of an exceptional set E of finite linear measure (or finite logarithmic measure). Then, for any $\alpha>1$, there exists $r_0>0$ such that $g(r)\leqslant h((\alpha r)$ (or $g(r)\leqslant h((r^\alpha))$ holds for all $r>r_0$.*

Theorem 1.1.10($\cos\pi\sigma$ theorem, see [7-9]) *Let $h(z)$ be an entire function with order $\sigma(h)=\sigma<\dfrac{1}{2}$, set*

$$A(r) = \inf_{|z|=r}\log|h(z)|, \quad B(r) = \sup_{|z|=r}\log|h(z)|.$$

If $\sigma<\alpha<1$, then

$$\underline{\log dens}\{r : A(r) > (\cos\pi\alpha)B(r)\} \geqslant 1-\frac{\sigma}{\alpha},$$

where the lower logarithmic density $\underline{\log dens}H$ of subset $H\subset(1,+\infty)$ is defined by

$$\underline{\log dens}H = \liminf_{r\to\infty}\left(\int_1^r (\chi_H(t)/t)dt\right)/\log r,$$

and the upper logarithmic density $\overline{\log dens}H$ of subset $H\subset(1,+\infty)$ is defined by

$$\overline{\log dens}H = \limsup_{r\to\infty}\left(\int_1^r (\chi_H(t)/t)dt\right)/\log r,$$

where $\chi_H(t)$ is the characteristic function of the set H.

Remark 1.1.4 *Let $h(z)$ be an entire function with lower order $\mu = \mu(h) < \frac{1}{2}$, and $\mu < \sigma = \sigma(h)$. If $\mu \leqslant \delta < \min\left(\sigma, \frac{1}{2}\right)$ and $\delta < \alpha < \frac{1}{2}$, then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r) > r^{\delta}\} \geqslant C(\sigma, \delta, \alpha),$$

where $C(\sigma, \delta, \alpha)$ is a positive constant only dependent on σ, δ, α.

By definitions of the logarithmic measure and the logarithmic density, we see that if the upper logarithmic density $\overline{\log dens} H > 0$, then the logarithmic measure $lmH = +\infty$.

1.2 Difference Analogue of the Lemma on the Logarithmic Derivative

The lemma on the logarithmic derivative states that outside of a possible small exceptional set

$$m\left(r, \frac{f'}{f}\right) = O(\log T(r, f) + \log r).$$

This is undoubtedly one of the most useful results of Nevanlinna theory, having a vast number of applications in the theory of meromorphic functions and in the theory of ordinary differential equations.

One major problem in the study of complex difference equations has so far been the lack of efficient tools, which can play roles similar to that played by relation the lemma on the logarithmic derivative for differential equations. This has meant that most results have had to be proved separately for each difference equation. This slows down the efforts to construct a coherent theory, and it may be one of the reasons why the theory of meromorphic solutions of complex difference equations is not as developed as the theory of differential equations.

Recently, there has been a renewed interested in the complex analytic properties of solutions of difference equations.

Halburd and Korhonen [85], Chiang and Feng [60] respectively gave difference analogues of the lemma on the logarithmic derivative. Halburd and Korhonen [85] proved the following Theorem 1.2.1 and Corollary 1.2.1. Chiang and Feng [60] proved the following Theorems 1.2.2–1.2.4 and Corollaries 1.2.2–1.2.5.

Theorem 1.2.1 *Let $f(z)$ be a nonconstant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^{\delta}}\right) \tag{1.2.1}$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

We need the following lemma for the proof of Theorem 1.2.1.

Lemma 1.2.1 *Let $f(z)$ be a meromorphic functionsuch that $f(0) \neq 0, \infty$ and let $c \in \mathbb{C}$. Then for all $\alpha > 1$, $\delta < 1$ and $r \geqslant 1$,*

$$m\left(r,\ \frac{f(z+c)}{f(z)}\right) \leqslant \frac{K(\alpha,\delta,c)}{r^{\delta}}\left(T(\alpha(r+|c|),\ f) + \log^{+}\frac{1}{|f(0)|}\right),$$

where

$$K(\alpha,\delta,c) = \frac{8|c|(3\alpha+1) + 8\alpha(\alpha-1)|c|^{\delta}}{\delta(1-\delta)(\alpha-1)^2 r^{\delta}}.$$

Proof Let $\{a_n\}$ denote the sequence of all zeros of f, and similarly let $\{b_m\}$ be the pole sequence of f, where $\{a_n\}$ and $\{b_m\}$ are listed according to their multiplicities and ordered by increasing modulus. By applying Poisson-Jensen formula with $s = \frac{\alpha+1}{2}(r+|c|)$, see, for instance, [89, Theorem 1.1], we obtain

$$\begin{aligned}\log\left|\frac{f(z+c)}{f(z)}\right| &= \int_0^{2\pi} \log|f(se^{i\theta})| \mathrm{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c} - \frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi} \\ &\quad + \sum_{|a_n|<s} \log\left|\frac{s(z+c-a_n)}{s^2-\overline{a}_n(z+c)}\frac{s^2-\overline{a}_n z}{s(z-a_n)}\right| \\ &\quad - \sum_{|b_m|<s} \log\left|\frac{s(z+c-b_m)}{s^2-\overline{b}_m(z+c)}\frac{s^2-\overline{b}_m z}{s(z-b_m)}\right| \\ &=: S_1(z) + S_2(z) - S_3(z). \qquad (1.2.2)\end{aligned}$$

Therefore, by denoting

$$E := \left\{\varphi \in [0,2\pi] : \left|\frac{f(re^{i\varphi}+c)}{f(re^{i\varphi})}\right| \geqslant 1\right\},$$

we have

$$\begin{aligned}m\left(r,\ \frac{f(z+c)}{f(z)}\right) &= \int_E \log\left|\frac{f(re^{i\varphi}+c)}{f(re^{i\varphi})}\right|\frac{d\varphi}{2\pi} \\ &\leqslant \int_0^{2\pi} \left|S_1(re^{i\varphi})\right| + \left|S_2(re^{i\varphi})\right| + \left|S_3(re^{i\varphi})\right|\frac{d\varphi}{2\pi}.\end{aligned}$$

We will now proceed to estimate each $\int_0^{2\pi} |S_j(re^{i\varphi})|\frac{d\varphi}{2\pi}$ separately. Since

$$\begin{aligned}|S_1| &= \left|\int_0^{2\pi} \log\left|f(se^{i\theta})\right| \mathrm{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-z-c)(se^{i\theta}-z)}\right)\frac{d\theta}{2\pi}\right| \\ &\leqslant \frac{2|c|s}{(s-r-|c|)^2}\int_0^{2\pi}\left|\log\left|f(se^{i\theta})\right|\right|\frac{d\theta}{2\pi} \\ &= \frac{2|c|s}{(s-r-|c|)^2}\left(m(s,\ f)+m\left(s,\ \frac{1}{f}\right)\right),\end{aligned}$$

we have

$$\int_0^{2\pi}\left|S_1(re^{i\varphi})\right|\frac{d\varphi}{2\pi} \leqslant \frac{4|c|s}{(s-r-|c|)^2}\left(T(s,\ f)+\log^+\frac{1}{|f(0)|}\right). \tag{1.2.3}$$

Next we consider the cases $j = 2, 3$ combined together. First, by denoting $\{q_k\} := \{a_n\}\bigcup\{b_m\}$ and using the fact that $|\log x| = \log^+ x + \log^+(1/x)$ for all $x > 0$, we have

$$\begin{aligned}\int_0^{2\pi}\left|S_2(re^{i\varphi})\right| + \left|S_3(re^{i\varphi})\right|\frac{d\varphi}{2\pi} \leqslant &\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\theta}-q_k}\right|\frac{d\theta}{2\pi} \\ &+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-q_k}\right|\frac{d\theta}{2\pi} \\ &+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{\overline{q}_k c}{s^2-\overline{q}_k(z+c)}\right|\frac{d\theta}{2\pi} \\ &+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{\overline{q}_k c}{s^2-\overline{q}_k z}\right|\frac{d\theta}{2\pi}. \end{aligned}\tag{1.2.4}$$

Second, for any $a \in \mathbb{C}$, and for all $\delta < 1$,

$$\int_0^{2\pi}\frac{d\theta}{|re^{i\theta}-a|^\delta} \leqslant 4\int_0^{\pi/2}\frac{d\theta}{|re^{i\theta}-|a||^\delta} \leqslant \frac{2\pi}{1-\delta}\frac{1}{r^\delta}$$

since $|re^{i\theta}-|a|| \geqslant r\theta\dfrac{2}{\pi}$ for all $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$. Therefore

$$\begin{aligned}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\theta}-a}\right|\frac{d\theta}{2\pi} &\leqslant \frac{1}{\delta}\int_0^{2\pi}\log^+\left(1+\left|\frac{c}{re^{i\theta}-a}\right|^\delta\right)\frac{d\theta}{2\pi} \\ &\leqslant \frac{1}{\delta}\int_0^{2\pi}\left|\frac{c}{re^{i\theta}-a}\right|^\delta\frac{d\theta}{2\pi} \leqslant \frac{|c|^\delta}{\delta(1-\delta)}\frac{1}{r^\delta}\ ,\end{aligned}\tag{1.2.5}$$

and similarly

$$\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-a}\right|\frac{d\theta}{2\pi} \leqslant \frac{|c|^\delta}{\delta(1-\delta)}\frac{1}{r^\delta}. \tag{1.2.6}$$