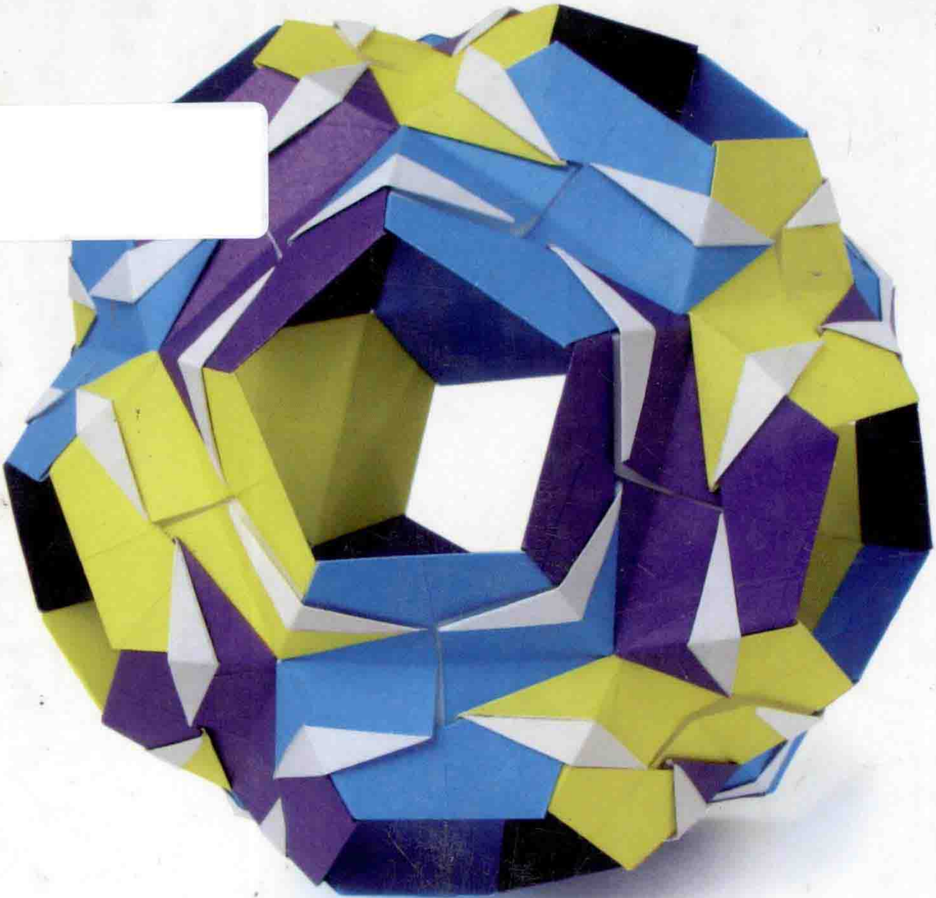

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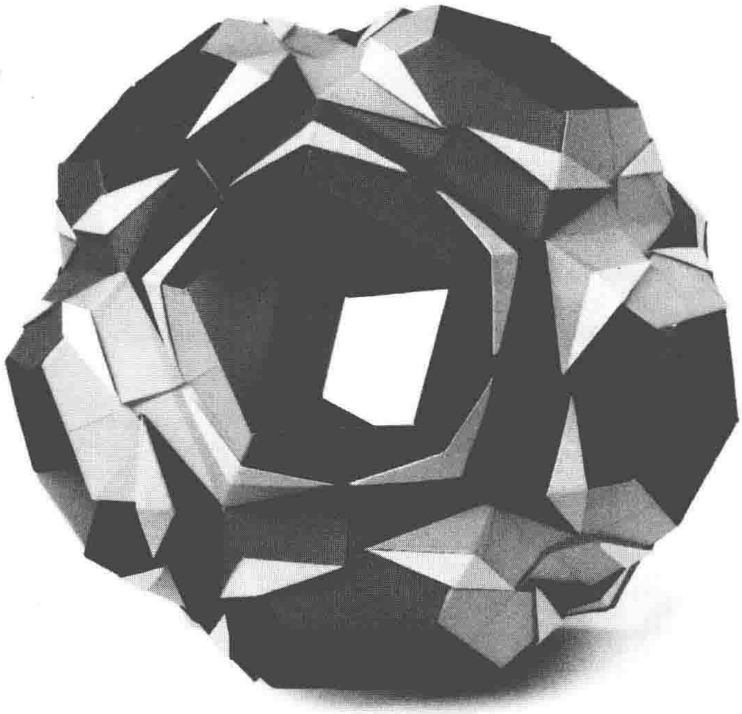

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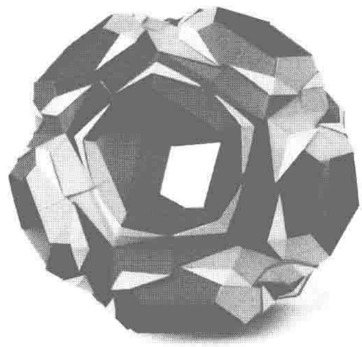

LECTURE NOTES ON

Algebraic Structure of Lattice-Ordered Rings

To Li, Cheng, and Elisa

Preface

This book is an introduction to the theory of lattice-ordered rings. It is suitable for graduate and advanced undergraduate students who have finished an abstract algebra class. It can also be used as a self-study book for one who is interested in the area of lattice-ordered rings.

The book mainly presents some foundations and topics in lattice-ordered rings. Since we concentrate on lattice orders, most results are stated and proved for such structures, although some of results are true for partially ordered structures. This book considers general lattice-ordered rings. However I have tried to compare results in general lattice-ordered rings with results in f-rings. Actually a lot of research work in general lattice-ordered rings is to generalize the results of f-rings. I have also tried to make the book self-contained and to give more details in the proofs of the results. Because of elementary nature of the book, some results are given without proofs. Certainly references are given for those results.

Chapter 1 consists of background information on lattice-ordered groups, vector lattices, and lattice-ordered rings and algebras. Those results are basic and fundamental. An important structure theory on lattice-ordered groups and vector lattices presented in Chapter 1 is the structure theory of lattice-ordered groups and vector lattices with a basis. Chapter 2 presents algebraic structure of lattice-ordered algebras with a distributive basis, which is a basis in which each element is a distributive element. Chapter 3 concentrates on positive derivations of lattice-ordered rings. This topic hasn't been systematically presented before and I have tried to present most of the important results in this area. In Chapter 4, some topics of general lattice-ordered rings are considered. Section 4.1 consists of some characterizations of lattice-ordered matrix rings with the entrywise order over lattice-ordered rings with positive identity element. Section 4.2 gives the algebraic structure of lattice-ordered rings with positive cycles. In general lattice-ordered rings, f-elements often play important roles on their structures. In Section 4.3 we present some result along this line. Section 4.4 is about extending lattice orders in an Ore domain to its quotient ring. In Section 4.5 we consider how to generalize results on lattice-ordered matrix algebras over totally ordered fields to lattice-ordered matrix algebras over totally ordered integral domains. Section 4.6 consists of some results on lattice-ordered rings in which the identity element may not be positive. In Section 4.7, all lattice orders on 2×2 upper triangular matrix algebras over a totally ordered field are constructed, and some results are given for higher dimension triangular matrix algebras. Finally in Chapter 5, properties and structure of ℓ-ideals of lattice-ordered rings with a positive identity elements are presented.

I would like to thank Dr. K.K. Phua, the Chairman and Editor-in-Chief of World Scientific Publishing, for inviting me to write this lecture notes volume. I also want to express my thanks to my colleague Ms. Judy Bergman, University of Houston-Clear Lake, who has kindly checked English usage and grammar of the book. I will certainly have full responsibility for mistakes in the book, and hopefully they wouldn't give the reader too much trouble to understand its mathematical contents.

Jingjing Ma

Houston, Texas, USA
December 2013

Contents

Chapter 1

Introduction to ordered algebraic systems

In this chapter, we introduce various ordered algebraic systems and present some basic and important properties of these systems.

1.1 Lattices

For a nonempty set A, a binary relation $\leq$ on A is called a *partial order* on A if the following properties are satisfied.

(1) (reflexivity) $a \leq a$ for all $a \in A$,
(2) (antisymmetry) $a \leq b$, $b \leq a$ implies $a = b$ for all $a, b \in A$,
(3) (transitivity) $a \leq b$, $b \leq c$ implies $a \leq c$, for all $a, b, c \in A$.

The set A under a partial order $\leq$ is called a *partially ordered set.* One may write $b \geq a$ to denote $a \leq b$, and $a < b$ (or $b > a$) to mean that $a \leq b$ and $a \neq b$. If either $a \leq b$ or $b \leq a$, then a and b are called *comparable*, otherwise a and b are called *incomparable.* A partial order $\leq$ on a set A is called a *total order* if any two elements in A are comparable. In the case that $\leq$ is a total order, A is called a *totally ordered set* or a *chain.* Suppose that two partial orders, $\leq$ and $\leq'$, are defined on the same set A. Then we say that $\leq'$ is an *extension* of $\leq$ if, for all $a, b \in A$, $a \leq b$ implies $a \leq' b$.

A partial order $\leq$ on A induces a partial order on any nonempty subset B of A, that is, for any $a, b \in B$, define $a \leq b$ in B if $a \leq b$ with respect to the original partial order of A. The induced partial order on B is denoted by the same symbol $\leq$.

For a subset B of a partially ordered set A an *upper bound* (*lower bound*) of B in A is an element $x \in A$ $(y \in A)$ such that $b \leq x$ $(b \geq y)$ for each $b \in B$. We may simply denote that $x \in A$ $(y \in A)$ is an upper (lower) bound of B by $B \leq x$ $(B \geq y)$. B is called *bounded* in A if B has both an upper bound and a lower bound in A. The set of all upper (lower) bounds of B in A is denoted by $U_A(B)$ ($L_A(B)$). If $B = \emptyset$, where $\emptyset$ denotes empty set, then $U_A(B) = L_A(B) = A$. An element $u \in B$ ($v \in B$) is called the *least element* (*greatest element*) of B if $u \leq b$ ($v \geq b$) for each $b \in B$. A subset B of a partially ordered set may not have a least (greatest) element, but if there exists one, then it is unique since partial orders are antisymmetric. An element $w \in B$ ($z \in B$) is called a *minimal element* (*maximal element*) in B if for any $b \in B$, $b \leq w$ ($b \geq z$) implies $b = w$ ($b = z$), that is, no element in B is strictly less (greater) than w (z). A subset of a partially ordered set may contain more than one minimal or maximal element.

Suppose that L is a partially ordered set with a partial order $\leq$. The $\leq$ is called a *lattice order* and L is called a *lattice* under $\leq$ if for any $a, b \in L$, the set $U_L(\{a,b\})$ has the least element and the set $L_L(\{a,b\})$ has the greatest element, namely, for any $a, b \in L$, the subset $\{a, b\}$ has the least upper bound and greatest lower bound that are denoted respectively by

$$a \vee b \quad \text{and} \quad a \wedge b$$

$a \vee b$ is also called the *sup* of a and b, and $a \wedge b$ is also called the *inf* of a and b. A nonempty subset B of a lattice L is called a *sublattice* of L if for any $a, b \in B$, $a \vee b, a \wedge b \in B$. A lattice L is called *distributive* if for all $a, b, c \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

and L is called *complete* if each subset of L has both an inf and a sup in L. In a lattice L, for any $a, b, c \in L$, by the definition of least upper bound and greatest lower bound, we have

$$a \vee (b \vee c) = (a \vee b) \vee c \quad \text{and} \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

This is true for any finitely many elements in L, and hence we just use $a_1 \vee \cdots \vee a_n$ and $a_1 \wedge \cdots \wedge a_n$ to denote the sup and inf of $a_1, \cdots, a_n$, respectively.

The following is an example that illustrates some concepts defined above. More examples may be found in the exercises of this chapter.

Example 1.1. For a given set A, let $P_A = \{B \mid B \text{ is a subset of } A\}$ be the power set of A. For two subsets B, C of A, define $B \leq C$ if $B \subseteq C$, where "$B \subseteq C$" means that B is a subset of C. Then $\leq$ is actually a lattice order and for any $B, C \in P_A$, $B \vee C = B \cup C$ and $B \wedge C = B \cap C$. Clearly $\emptyset$ is the least element of P_A and A is the greatest element of P_A. Moreover, P_A is a distributive and complete lattice (Exercise 3).

If A contains more than one element, then P_A is not a totally ordered set since for two different elements $a, b \in A$, the sets $\{a\}$ and $\{b\}$ are not comparable. Also the subset $B = \{\{a\}, \{b\}\}$ of P_A has no least and greatest element, and each element in B is a minimal element and a maximal element since $\{a\}$ and $\{b\}$ are not comparable.

This is a suitable place to state Zorn's lemma, which is equivalent to Axiom of Choice. For the proof and other equivalent forms of the lemma, see [Steinberg (2010)].

Theorem 1.1 (Zorn's Lemma). *Let A be a nonempty partially ordered set. If each subset of A which is a chain has an upper bound in A, then A contains a maximal element.*

1.2 Lattice-ordered groups and vector lattices

In this section we introduce partially ordered groups, lattice-ordered groups, vector lattices, and consider some basic properties of those ordered algebraic systems. We will always use addition to denote group operation although it may not be commutative. Certainly for a vector lattice, the addition on it is commutative.

1.2.1 *Definitions, examples, and basic properties*

Definition 1.1. A *partially ordered group* G is a group and a partially ordered set under a partial order $\leq$ such that G satisfies the following monotony law: for any $a, b \in G$,

$$a \leq b \Rightarrow c + a \leq c + b \text{ and } a + c \leq b + c \text{ for all } c \in G.$$

A partially ordered group G is a *lattice-ordered group* (*ℓ-group*) if the partial order is a lattice order, and G is a *totally ordered group* (*o-group*) if the partial order is a total order.

In a partially ordered group G, an element g is called *positive* if $g \geq 0$, where 0 is the identity element of G, and g is called *strictly positive* if $g > 0$. The set $G^+ = \{g \in G \mid g \geq 0\}$ is called the *positive cone* of G, and define $-G^+ = \{g \in G \mid -g \in G^+\} = \{g \in G \mid g \leq 0\}$, which is called *negative cone* of G. G^+ is a normal subsemigroup of G containing 0, but no other element along with its inverse, as shown in the following result. From the following two theorems, positive cones characterize partially ordered groups.

Theorem 1.2. *For a partially ordered group G, the positive cone G^+ satisfies the following three conditions:*

(1) $G^+ + G^+ \subseteq G^+$,
(2) $g + G^+ + (-g) \subseteq G^+$, *for all* $g \in G$,
(3) $G^+ \cap -G^+ = \{0\}$.

Proof. (1) Let $g, f \in G^+$. Then $0 \leq f \leq g + f$, so $0 \leq g + f$. Thus $g + f \in G^+$.

(2) Let $f \in G^+$. Then $0 = g+(-g) \leq g+f+(-g)$, so $g+f+(-g) \in G^+$.

(3) Clearly $0 \in G^+ \cap -G^+$. Suppose that $g \in G^+ \cap -G^+$. Then $g \geq 0$ and $-g \geq 0$, so $g \geq 0$ and $g \leq 0$, and hence $g = 0$. □

Theorem 1.3. *Let G be a group and P be a subset of G which satisfies the following three conditions:*

(1) $P + P \subseteq P$,
(2) $g + P + (-g) \subseteq P$ *for all* $g \in G$,
(3) $P \cap -P = \{0\}$, *where* $-P = \{g \in G \mid -g \in P\}$.

For any $a, b \in G$, define $a \leq b$ if $b - a \in P$. Then $\leq$ is a partial order on G and G becomes a partially ordered group with the positive cone P.

Proof. For any $a \in G$, $a - a = 0 \in P$ implies $a \leq a$, so $\leq$ is reflexive. Suppose that for $a, b \in G$, $a \leq b$ and $b \leq a$, then $b-a, a-b \in P$, so $b-a \in P$ and $b - a = -(a - b) \in -P$. Thus $b - a = 0$ by (3), and hence $a = b$, so $\leq$ is antisymmetric. Now assume that $a \leq b$ and $b \leq c$ for $a, b, c \in G$. Then $b - a, c - b \in P$, so by (1) $c - a = (c - b) + (b - a) \in P$. Thus $a \leq c$, so $\leq$ is transitive. Suppose that $a \leq b$ for $a, b \in G$ and $g \in G$. Then from $b - a \in P$ and (2),

$$(g + b) - (g + a) = g + (b - a) + (-g) \in P,$$

so $g + a \leq g + b$. Also

$$(b + g) - (a + g) = b + g - g - a = b - a \in P,$$

so $a + g \leq b + g$. Therefore G is a partially ordered group with respect to the partial order $\leq$. Clearly $G^+ = \{g \in G \mid g \geq 0\} = P$. □

Theorem 1.4. *Suppose that G is a partially ordered group with the positive cone P.*

(1) G is an ℓ-group if and only if $G = \{a - b \mid a, b \in P\}$ and P is a lattice under the induced partial order from G.
(2) G is a totally ordered group if and only if $G = P \cup -P$.

Proof. (1) Suppose that G is an ℓ-group. For $g \in G$, let $f = g \wedge 0$. Then $-f \in P$ and $g - f \in P$. Since $g = (g - f) - (-f)$, $G = \{a - b \mid a, b \in P\}$. It is clear that for any $a, b \in P$, $a \vee b, a \wedge b \in P$. Conversely, suppose that $G = \{a - b \mid a, b \in P\}$ and P is a lattice with respect to the induced partial order from G. For any $g \in G$, let $g = x - y$, $x, y \in P$. Suppose that $z = x \vee y \in P$. We claim that $g \vee 0 = z - y$ in G. It is clear that $z - y \geq 0, g$. Suppose that $u \in G$ and $u \geq g, 0$. Then $u + y \geq x, y$ and $u + y \in P$, so $u + y \geq z$. Then it follows that $u \geq z - y$, and hence $g \vee 0 = z - y$ in G. Similarly to show that $g \wedge 0$ exists in G. Generally for any $g, f \in G$, it is straightforward to check that

$$g \vee f = [(g - f) \vee 0] + f \text{ and } g \wedge f = [(g - f) \wedge 0] + f$$

(Exercise 5). Therefore G is a lattice, so G is an ℓ-group.

(2) If $G = P \cup -P$, then for any $g, f \in G$, either $g - f \in P$ or $-P$, and hence $g \geq f$ or $g \leq f$. Thus G is a total order. The converse is clear. □

A partially ordered group is called *directed* if each element is a difference of two positive elements. An ℓ-group is directed by Theorem 1.4(1). However a partially ordered group which is directed may not be an ℓ-group as shown in Example 1.2(3). A partially ordered group G is said to be *Archimedean* if for any $a, b \in G^+$, $na \leq b$ for all $n \in \mathbb{Z}^+$ implies $a = 0$, where $\mathbb{Z}^+$ is the set of all positive integers.

In this book we often use notation (G, P) to denote a partially ordered group or an ℓ-group with the positive cone P.

We illustrate partially ordered groups and ℓ-groups by a few examples. P will always denote the positive cone of a partially ordered group.

Example 1.2.

(1) Let G be the additive group of $\mathbb{Z}$ or $\mathbb{Q}$, or $\mathbb{R}$ with the usual order between real numbers. Then G is an Archimedean totally ordered group.
(2) Consider the group direct product $\mathbb{R} \times \mathbb{R}$. Let (x, y) belong to P if either $y > 0$ or $y = 0$ and $x \geq 0$. Then $\mathbb{R} \times \mathbb{R}$ is a totally ordered group which is not Archimedean since for any $n \in \mathbb{Z}^+$, $n(1, 0) \leq (0, 1)$.
(3) Consider $\mathbb{R} \times \mathbb{R}$ again. Define $(x, y) \in P$ if $x > 0$ and $y > 0$, or $(x, y) = (0, 0)$. Then $\mathbb{R} \times \mathbb{R}$ is an Archimedean partially ordered group but not an ℓ-group. For instance, $(1, 0)$ and $(0, 0)$ have no least upper

bound. We leave the verification of this fact as an exercise to the reader (Exercise 6). We note that for any $(x, y) \in \mathbb{R}\times\mathbb{R}$, $(x, y) = (x, 0)+(0, y)$, and $(x, 0), (0, y)$ are either positive or negative, so (x, y) can be written as a difference of two positive elements. Thus this partially ordered group is directed.

Since in this book, we concentrate on lattice orders, in the following we only prove some basic properties of ℓ-groups.

Theorem 1.5. *Let G be an ℓ-group.*

(1) For all $a, b, c, d \in G$, $c + (a \vee b) + d = (c + a + d) \vee (c + b + d)$, $c + (a \wedge b) + d = (c + a + d) \wedge (c + b + d)$.
(2) For all $a, b \in G$, $-(a \vee b) = (-a) \wedge (-b)$, $-(a \wedge b) = (-a) \vee (-b)$.
(3) As a lattice, G is distributive.
(4) For all $a, b \in G$, $a - (a \wedge b) + b = a \vee b$. If G is commutative, then $a + b = (a \wedge b) + (a \vee b)$, for all $a, b \in G$.
(5) If $na \geq 0$ for some positive integer n, then $a \geq 0$.
(6) If $x, y_1, \cdots, y_n$ are positive elements such that $x \leq y_1 + \cdots + y_n$, then $x = x_1+\cdots+x_n$ for some positive elements $x_1, \cdots, x_n$ with $x_i \leq y_i, i = 1, \cdots, n$.
(7) If $x, y_1, \cdots, y_n$ are positive elements, then $x \wedge (y_1 + \cdots + y_n) \leq (x \wedge y_1) + \cdots + (x \wedge y_n)$.

Proof. (1) From $a\vee b \geq a, b$, we have $c+(a\vee b)+d \geq (c+a+d), (c+b+d)$, so

$$c + (a \vee b) + d \geq (c + a + d) \vee (c + b + d).$$

On the other hand, $(c+a+d), (c+b+d) \leq (c+a+d) \vee (c+b+d)$ implies

$$a, b \leq -c + (c + a + d) \vee (c + b + d) + (-d),$$

and hence

$$a \vee b \leq -c + (c + a + d) \vee (c + b + d) + (-d).$$

Therefore $c + (a \vee b) + d \leq (c + a + d) \vee (c + b + d)$. We conclude that $c+(a\vee b)+d = (c+a+d)\vee(c+b+d)$. Similarly we have $c+(a\wedge b)+d = (c + a + d) \wedge (c + b + d)$.

(2) We have

$$a, b \leq a \vee b \Rightarrow -(a \vee b) \leq -a, -b \Rightarrow -(a \vee b) \leq -a \wedge -b,$$

and

$$-a \wedge -b \leq -a, -b \Rightarrow a, b \leq -(-a \wedge -b) \Rightarrow a \vee b \leq -(-a \wedge -b),$$