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ABSOLUTE MEASURABLE SPACES

Togo Nishiura

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Absolute Measurable Spaces

Absolute measurable space and absolute null space are very old topological notions, developed from descriptive set theory, topology, Borel measure theory and analysis. This monograph systematically develops and returns to the topological and geometrical origins of these notions. Motivating the development of the exposition are the action of the group of homeomorphisms of a space on Borel measures, the Oxtoby–Ulam theorem on Lebesgue-like measures on the unit cube, and the extensions of this theorem to many other topological spaces. Existence of uncountable absolute null space, extension of the Purves theorem, and recent advances on homeomorphic Borel probability measures on the Cantor space are among the many topics discussed. A brief discussion of set-theoretic results on absolute null space is also given.

A four-part appendix aids the reader with topological dimension theory, Hausdorff measure and Hausdorff dimension, and geometric measure theory. The exposition will suit researchers and graduate students of real analysis, set theory and measure theory.

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Preface

This book is about absolute measurable spaces. What is an absolute measurable space and why study them?

To answer the first question, an absolute measurable space, simply put, is a separable metrizable space X with the property that every topological embedding of X into any separable metrizable space Y results in a set that is *μ-measurable for every continuous, complete, finite Borel measure μ on Y*. Of course, only Borel measures are considered since the topology of Y must play a role in the definition.

For an answer to the second question, observe that the notion of absolute measurable space is a topological one in the spirit of many other notions of "absolute" such as absolute Borel space, absolute G_δ space, absolute retract and many more. As the definition is topological, one is led to many topological questions about such spaces. Even more there are many possible geometric questions about such spaces upon assigning a metric to the space. Obviously, there is also a notion of "absolute null space"; these spaces are those absolute measurable spaces for which all topological copies have μ measure equal to 0. Absolute null spaces are often called "universal measure zero sets" and have been extensively studied. The same topological and geometric questions can be investigated for absolute null spaces. It is well-known that absolute Borel spaces are absolute measurable spaces. More generally, so are analytic and co-analytic spaces. Many topological and geometric questions have already been investigated in the literature for absolute Borel spaces and analytic spaces. The challenge is to prove or disprove analogues of these known results in the context of absolute measurable spaces.

It is clear that absolute measurable spaces are invariant under Borel isomorphism (Borel measurable bijection whose inverse is also Borel measurable). Consequently, each absolute measurable space will correspond to an absolute measurable subspace of the real line $\mathbb{R}$. It would be tempting to investigate only absolute measurable spaces contained in $\mathbb{R}$, which has been extensively done. This would be fine if one is interested only in, say, measure theoretic or set theoretic properties of absolute measurable spaces, but clearly inadequate if one is interested in topological or geometric structures since they may not be preserved by Borel isomorphisms. The emphasis of the book is on topological and geometric properties associated with absolute measurable spaces. Homeomorphisms will be emphasized for topological structures. For geometric structures, one must have a metric assigned to the separable metrizable space – bi-Lipschitzian maps will replace homeomorphisms.

There is a second notion called "universally measurable sets." This notion fixes a space X and considers the collection of all subsets of X that are μ-measurable for every continuous, complete, finite Borel measure μ on X. Obviously a subset of X that is an absolute measurable space is a universally measurable set in X. But a universally measurable set in a space X need not be an absolute measurable space – indeed, for a non-Lebesgue measurable set X of $\mathbb{R}$, the set X itself is a universally measurable set in X that is not an absolute measurable space. It is easily seen that X is an absolute measurable space if and only if every universally measurable set of X is an absolute measurable space.

An extensive literature exists concerning the notions of absolute measurable space and universally measurable set. The 1982 survey article **[18]**, written by J. B. Brown and G. V. Cox, is devoted to a large number of classes of "singular" spaces among which is the class of absolute null spaces. Their article is essentially a broad ranging summary of the results up to that time and its coverage is so ambitious that a systematic development from the basics of real analysis and topology has not been presented. There are two other survey articles that are devoted to set theoretic results on certain singular sets. From the set theoretic point of view only subsets of the real line needed to be considered. The first article is a 1984 survey about such subsets by A. W. Miller **[110]** and the second is his 1991 update **[111]**. Absolute measurable spaces and absolute null spaces have appeared also in probability theory – that is, probability theory based on abstract measurable spaces $(X, \mathfrak{A})$ in which metrics are induced on X by imposing conditions on the σ-algebra $\mathfrak{A}$ of measurable sets. Obviously this approach to the notion of absolute measurable space concentrates on probability concepts and does not investigate topological and geometric properties. In 1984, R. M. Shortt investigated metric properties from the probability approach in **[139]** (announced in 1982 **[138]**). Also in non-book form are two articles that appeared much earlier in 1937; one is a commentary by S. Braun and E. Szpilrajn in collaboration with K. Kuratowski that appeared in the "Annexe" **[15]** to the new series of the *Fundamenta Mathematicae* and the other is a fundamental one by Szpilrajn-Marczewski **[152]** that contains a development of the notions of absolute measurable space and universally measurable subsets of a metric space with applications to singular sets. Years have passed since the two articles were written.

The book sets aside many singular sets whose definitions depend on a chosen metric; fortunately, the definition of the Lebesgue measure on the real line depends only on the arithmetic structure of the real number system and is metric independent. This setting aside of metric-dependent singular set theory permits a systematic development, beginning with the basics of topology and analysis, of absolute measurable space and universally measurable sets in a separable metrizable space. Two themes will appear. One deals with the question of the possibility of strengthening theorems by replacing absolute Borel spaces in the hypothesis of known theorems with absolute measurable spaces. The other is an investigation of the possibility of extending topological properties or geometric properties of universally measurable sets in $\mathbb{R}$ to absolute measurable spaces X other than $\mathbb{R}$. The first question is complicated by the following unresolved set theoretic question **[110]** due to R. D. Mauldin. Note that there are $\mathfrak{c}$ Borel sets in $\mathbb{R}$.

(Mauldin) *What is the cardinality of the collection of all absolute measurable subspaces of the real line* $\mathbb{R}$*?* In particular, *are there always more than* $\mathfrak{c}$ *absolute measurable subspaces of* $\mathbb{R}$*?*

The cardinality of absolute null spaces plays a role in Mauldin's question since an absolute measurable space is not necessarily the symmetric difference of an absolute Borel space and an absolute null space.

There are six chapters plus a four-part appendix. The first chapter is a systematic development of the notions of absolute measurable space and absolute null space. Clearly countable separable metrizable spaces are always absolute null spaces. Solutions of the question of the existence, under the usual axioms of set theory, of uncountable absolute null spaces are presented.

The second chapter is a systematic development of the notion of universally measurable sets in a separable metrizable space X. The concept of positive measures (loosely speaking, $\mu(U) > 0$ whenever U is a nonempty open set) is introduced. This concept leads naturally to the operation called *positive closure* which is a topological invariant. Of particular interest is the example [0, 1] and HOMEO([0, 1]), the group of all homeomorphisms of [0, 1]. It is a classical result that the collection of all universally measurable sets in [0, 1] is generated by the Lebesgue measure λ on [0, 1] and HOMEO([0, 1]). Even more, it is known that the collection of all positive, continuous, complete, finite Borel measures on [0, 1] is generated by λ and HOMEO([0, 1]).

The topological project of replacing the space [0, 1] with other absolute measurable spaces is the focus of the third chapter. This project, which addresses the second of the two above mentioned classical results, leads naturally to the Oxtoby–Ulam theorem and its many generalizations. The Oxtoby–Ulam theorem does not generalize to the Cantor space $\{0, 1\}^{\mathbb{N}}$. Fortunately there is a Radon–Nikodym derivative version of the Oxtoby–Ulam theorem which includes the Cantor space and allows the introduction of analysis into the book.

There are many results in analysis on functions $f : \mathbb{R} \to \mathbb{R}$ in the context of universally measurable sets in $\mathbb{R}$. Chapter 4 is devoted to the question of the replacement of the domain or the range of f by absolute measurable spaces. The usual approach of using Borel isomorphisms does not necessarily apply to the task at hand. But the results of Chapter 3 can be applied.

Chapter 5 is devoted to geometric properties of universally measurable sets in $\mathbb{R}^n$ – in particular, the Hausdorff measure and Hausdorff dimension of absolute null spaces. Results, due to O. Zindulka, that sharpen the classical inequalities between Hausdorff dimension and topological dimension form the main part of the chapter.

Finally, Chapter 6 is a short discussion of the set theoretic aspect of absolute measurable spaces. The literature on this aspect is quite extensive. Only a brief survey is given of the use of the continuum hypothesis and the Martin axiom in the book. Of particular interest is the topological dimension of absolute null spaces. Surprisingly, the result, due to Zindulka, depends on set axioms.

Appendix A collects together the needed descriptive set theoretic results and measure theoretic results that are used in the book. Developing notational consistency is also an objective of this part. A proof of the Purves theorem is also presented since it is extended to include universally measurable sets and universally null sets in Chapter 2.

Appendix B is a brief development of universally measurable sets and universally null sets from the measure theoretic and probability theoretic point of view, which reverses our "Borel sets lead to probability measures" to "probability measures lead to Borel sets." This reversal places emphasis on Borel isomorphism and not on homeomorphism; consequently, topological and geometrical questions are not of interest here.

Appendix C concerns Cantor spaces (metrizable spaces that are nonempty, compact, perfect and totally disconnected). Cantor spaces have many realizations, for example, k^ω, where k is a finite set with $\mathrm{card}(k) > 1$. The homeomorphism equivalence classes of positive, continuous, complete Borel probability measures on a topological Cantor space are not very well understood. Even the Bernoulli measures on k^ω are not completely understood. Extensive investigations by many authors have been made for $\mathrm{card}(k) = 2$. In this case a weaker equivalence relation introduces a connection to polynomials with coefficients in $\mathbb{Z}$. These polynomials are special Bernstein polynomials found in classical approximation theory. Recent results of R. Dougherty, R. D. Mauldin and A. Yingst [**47**] and T. D. Austin [**6**] are discussed and several examples from the earlier literature are given. The E. Akin approach of introducing topological linear order into the discussion of Cantor spaces is also included.

Finally, Appendix D is a brief survey of Hausdorff measure, Hausdorff dimension, and topological dimension. These concepts are very important ones in the book. Zindulka's new proof of the classical relationship between the Hausdorff and topological dimensions is given.

The book is somewhat self-contained; many complete proofs are provided to encourage further investigation of absolute measurable spaces.

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First I want to thank Jack Brown for providing many of the references in his possession and for the help he gave me in the initial phase of the writing of the book. Next, thanks go to R. D. Mauldin for the many conversations we had on various major topics presented in the book; he has written widely on these topics and his knowledge of the literature surrounding the Oxtoby-motivated theorems greatly improved the exposition. Also, I wish to acknowledge, with thanks, several helpful conversations with A. W. Miller. I want to thank K. P. Hart for his help with certain aspects of the development of the material on Martin's axiom. Also, thanks go to P. Mattila for help with geometric measure theory. Two authors, E. Akin and O. Zindulka, were very kind in providing me with their works in prepublication form. Akin's works on the homeomorphism group of the Cantor space and his Oxtoby-motivated theorems were very influential, especially his construction of the bi-Lipschitzian equivalence of Radon–Nikodym derivatives associated with probability measures on the Cantor space; Zindulka's dimension theoretic results motivated the final two chapters and the appendix. Finally I wish to acknowledge the help and support provided by the Library and the Department of Mathematics and Computer Science of Dickinson College.

Contents

1

The absolute property

A measure space $\mathrm{M}(X,\mu)$ is a triple $(X,\mu,\mathfrak{M}(X,\mu))$, where μ is a countably additive, nonnegative, extended real-valued function whose domain is the σ-algebra $\mathfrak{M}(X,\mu)$ of subsets of a set X and satisfies the usual requirements. A subset M of X is said to be μ-measurable if M is a member of the σ-algebra $\mathfrak{M}(X,\mu)$.

For a separable metrizable space X, denote the collection of all Borel sets of X by $\mathfrak{B}(X)$. A measure space $\mathrm{M}(X,\mu)$ is said to be *Borel* if $\mathfrak{B}(X) \subset \mathfrak{M}(X,\mu)$, and if $M \in \mathfrak{M}(X,\mu)$ then there is a Borel set B of X such that $M \subset B$ and $\mu(B) = \mu(M)$.[1] Note that if $\mu(M) < \infty$, then there are Borel sets A and B of X such that $A \subset M \subset B$ and $\mu(B \setminus A) = 0$.

Certain collections of measure spaces will be referred to often – for convenience, two of them will be defined now.

NOTATION 1.1 (MEAS; $\mathrm{MEAS}^{\mathrm{finite}}$). *The collection of all complete, σ-finite Borel measure spaces* $\mathrm{M}(X,\mu)$ *on all separable metrizable spaces X will be denoted by* MEAS. *The subcollection of* MEAS *consisting of all such measures that are finite will be denoted by* $\mathrm{MEAS}^{\mathrm{finite}}$.[2]

In the spirit of absolute Borel space, the notion of absolute measurable space will be defined in terms of μ-measurability with respect to all Borel measure spaces $\mathrm{M}(Y,\mu)$ in the collection MEAS. After the notion of absolute measurable space has been developed, the notion of absolute 0-measure space – more commonly known as absolute null space – is defined and developed. Two early solutions to the question of the existence of uncountable absolute null spaces are presented. They use the notion of m-convergence introduced by F. Hausdorff [**73**]. A more recent example, due to E. Grzegorek [**68**], that has other properties is also developed. The theorems due to S. Plewik [**127**, Lemma] and to I. Recław [**130**] will conclude the discussion of existence.

1.1. Absolute measurable spaces

DEFINITION 1.2. *Let X be a separable metrizable space. Then X is called an* absolute measurable space *if, for every Borel measure space* $\mathrm{M}(Y,\mu)$ *in* MEAS*, it is*

[1] Such measures are often called *regular* Borel measures. We have dropped the modifier regular for convenience.

[2] See also equations (A.4) and (A.5) on page 187 of Appendix A.

true that every topological copy M of X that is contained in Y is a member of the σ-algebra $\mathfrak{M}(Y,\mu)$. The collection of all absolute measurable spaces will be denoted by abMEAS.

Obviously, the notion of absolute measurable space is invariant under homeomorphisms. Hence it would be appropriate to define the notion of topological equivalence for Borel measure spaces on separable metrizable spaces. In order to do this we need the following definition of measures $f_{\#}\mu$ induced by measurable maps f.

DEFINITION 1.3 ($f_{\#}\mu$). *Let X and Y be separable metrizable spaces, let $\mathrm{M}(X,\mu)$ be a σ-finite Borel measure space, and let $f : X \to Y$ be a μ-measurable map. A subset M of Y is said to be* ($f_{\#}\mu$)-measurable *if there exist Borel sets A and B in Y such that $A \subset M \subset B$ and $\mu(f^{-1}[B \setminus A]) = 0$.*

It is clear that $\mathrm{M}(f_{\#}\mu, Y)$ is a complete, finite Borel measure on Y whenever $\mu(X) < \infty$, and that $\mathrm{M}(f_{\#}\mu, Y)$ is complete and σ-finite whenever f is a homeomorphism of X into Y and μ is σ-finite.[3]

DEFINITION 1.4. *σ-finite Borel measure spaces $\mathrm{M}(X,\mu)$ and $\mathrm{M}(Y,\nu)$ are said to be* topologically equivalent *if there is a homeomorphism h of X onto Y such that $h_{\#}\mu(B) = \nu(B)$ whenever $B \in \mathfrak{B}(Y)$.*

The last definition does not require that the Borel measure spaces be complete – but $h_{\#}$ does induce complete measure spaces. Hence the identity homeomorphism id_X of a space X yields a complete Borel measure space $\mathrm{M}(\mathrm{id}_{X\#}\,\mu, X)$, indeed, the measure completion of $\mathrm{M}(\mu, X)$.

It is now evident that there is no loss in assuming that the absolute measurable space X is contained in the Hilbert cube $[0,1]^{\mathbb{N}}$ for topological discussions of the notion of absolute measurable space.

1.1.1. Finite Borel measures. Often it will be convenient in discussions of absolute measurable spaces to deal only with finite Borel measure spaces rather than the more general σ-finite ones – that is, the collection $\mathrm{MEAS}^{\mathrm{finite}}$ rather than MEAS. The following characterization will permit us to do this.

THEOREM 1.5. *A separable metrizable space X is an absolute measurable space if and only if, for every Borel measure space $\mathrm{M}(Y,\mu)$ in $\mathrm{MEAS}^{\mathrm{finite}}$, it is true that every topological copy M of X that is contained in Y is a member of $\mathfrak{M}(Y,\mu)$.*

PROOF. Clearly, if a space X is an absolute measurable space, then it satisfies the condition given in the theorem. So suppose that X satisfies the condition of the theorem. Let $\mathrm{M}(Y,\mu)$ be a σ-finite Borel measure space. There is a finite Borel measure space $\mathrm{M}(Y,\nu)$ such that the σ-algebra equality $\mathfrak{M}(Y,\mu) = \mathfrak{M}(Y,\nu)$ holds (see Section A.5 of Appendix A). So $M \in \mathfrak{M}(Y,\mu)$, hence X is an absolute measurable space. □

[3] See Appendix A for more on the operator $f_{\#}$.

1.1.2. Continuous Borel measure spaces. Later it will be necessary to consider the smaller collection of all continuous Borel measure spaces.[4] If this smaller collection is used in Definition 1.2 above, it may happen that more spaces become absolute measurable spaces. Fortunately, this will not be the case because of our assumption that all measure spaces in MEAS are σ-finite. Under this assumption, for a measure μ, the set of points x for which $\mu(\{x\})$ is positive is a countable set. As continuous Borel measures have measure zero for every countable set, the collection of absolute measure spaces will be the same when one considers the smaller collection of all continuous, complete, σ-finite Borel measure spaces. The following notation will be used.

Notation 1.6 ($\mathrm{MEAS}^{\mathrm{cont}}$). *The collection of all continuous, complete, σ-finite Borel measure spaces* $\mathrm{M}(X, \mu)$ *on all separable metrizable spaces X is denoted by* $\mathrm{MEAS}^{\mathrm{cont}}$. *That is,*

$$\mathrm{MEAS}^{\mathrm{cont}} = \{\, \mathrm{M}(X, \mu) \in \mathrm{MEAS} \colon \mathrm{M}(X, \mu) \text{ is continuous} \,\}. \tag{1.1}$$

1.1.3. Elementary properties. Let us describe some properties of absolute measurable spaces. Clearly, each absolute Borel space is an absolute measurable space. The M. Lavrentieff theorem (Theorem A.2) leads to a characterization of absolute Borel spaces. This characterization yields the following useful characterization of absolute measurable spaces.

Theorem 1.7. *Let X be a separable metrizable space. The following statements are equivalent.*

(1) *X is an absolute measurable space.*
(2) *There exists a completely metrizable space Y and there exists a topological copy M of X contained in Y such that $M \in \mathfrak{M}(Y, \nu)$ for every complete, finite Borel measure space* $\mathrm{M}(Y, \nu)$.
(3) *For each complete, finite Borel measure space* $\mathrm{M}(X, \mu)$ *there is an absolute Borel space A contained in X with $\mu(X \setminus A) = 0$.*

Proof. It is clear that the first statement implies the second.

Assume that the second statement is true and let $h \colon X \to M$ be a homeomorphism. Then $\mathrm{M}(Y, h_{\#}\mu)$ is a complete Borel measure space in $\mathrm{MEAS}^{\mathrm{finite}}$. There exists a Borel set A' such that $A' \subset M$ and $h_{\#}\mu(M \setminus A') = 0$. As Y is a completely metrizable space, the space A' is an absolute Borel space. The restricted measure space $\mathrm{M}(M, (h_{\#}\mu)|M)$ is complete and is topologically equivalent to $\mathrm{M}(X, \mu)$. So $\mu(X \setminus A) = 0$, where $A = h^{-1}[A']$. As A' is an absolute Borel space, we have A is an absolute Borel space; hence statement (3) follows.

Finally let us show statement (3) implies statement (1). Let Y be a space and let M be a topological copy of X contained in Y. Suppose that $\mathrm{M}(Y, \mu)$ is complete and finite. Then $\mathrm{M}(M, \mu|M)$ is also complete and finite. It is easily seen that statement (3) is invariant under topological equivalence of Borel measure spaces. Hence $\mathrm{M}(M, \mu|M)$

[4] See Appendix A, page 187, for the definition of continuous Borel measure space.

also satisfies statement (3). There is an absolute Borel space A such that $A \subset M$ and $(\mu|M)(M \setminus A) = 0$. As $\mu^*(M \setminus A) = (\mu|M)(M \setminus A) = 0$, we have $M \setminus A \in \mathfrak{M}(Y, \mu)$, whence $M = (M \setminus A) \cup A$ is in $\mathfrak{M}(Y, \mu)$. □

1.1.4. σ-ring properties. As an application of the above theorem, let us investigate a σ-ring property of the collection abMEAS of all absolute measurable spaces. We begin with closure under countable unions and countable intersections.

PROPOSITION 1.8. *If $X = \bigcup_{i=1}^{\infty} X_i$ is a separable metrizable space such that each X_i is an absolute measurable space, then X and $\bigcap_{i=1}^{\infty} X_i$ are absolute measurable spaces.*

PROOF. Let Y be a completely metrizable extension of X and ν be a complete, finite Borel measure on Y. Then $X_i \in \mathfrak{M}(Y, \nu)$ for every i. Hence $X \in \mathfrak{M}(Y, \nu)$ and $\bigcap_{i=1}^{\infty} X_i \in \mathfrak{M}(Y, \nu)$. Theorem 1.7 completes the proof. □

PROPOSITION 1.9. *If $X = X_1 \cup X_2$ is a separable metrizable space such that X_1 and X_2 are absolute measurable spaces, then $X_1 \setminus X_2$ is an absolute measurable space.*

PROOF. Let Y be a completely metrizable extension of X and ν be a complete, finite Borel measure on Y. Then $X_i \in \mathfrak{M}(Y, \nu)$ for $i = 1, 2$. Hence $X_1 \setminus X_2$ is in $\mathfrak{M}(Y, \nu)$. Theorem 1.7 completes the proof. □

The σ-ring property of the collection abMEAS has been established. The next proposition follows from the ring properties.

PROPOSITION 1.10. *If X is a Borel subspace of an absolute measurable space, then X is an absolute measurable space.*

PROOF. Let Y be an absolute measurable space that contains X as a Borel subspace. Let Y_0 be a completely metrizable extension of Y. There exists a Borel subset B of Y_0 such that $X = Y \cap B$. As Y_0 is a completely metrizable space, we have that B is an absolute Borel space, whence an absolute measurable space. The intersection of the spaces Y and B is an absolute measurable space. □

1.1.5. Product properties. A finite product theorem for absolute measurable spaces is easily shown.

THEOREM 1.11. *A nonempty, separable, metrizable product space $X_1 \times X_2$ is also an absolute measurable space if and only if X_1 and X_2 are nonempty absolute measurable spaces.*

The proof is a consequence of the following proposition whose proof is left to the reader as it follows easily from Lemma A.34 in Appendix A.

PROPOSITION 1.12. *Let $\mathrm{M}(Y_1 \times Y_2, \mu)$ be a complete, σ-finite Borel measure space. If X_1 is an absolute measurable subspace of Y_1, then $X_1 \times X_2$ is μ-measurable.*