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Function Theory on Symplectic Manifolds

Preface

The symplectic revolution of the 1980s gave rise to the discovery of surprising rigidity phenomena involving symplectic manifolds, their subsets, and their diffeomorphisms. These phenomena have been detected with the help of a variety of novel powerful methods, including Floer theory, a version of Morse theory on the loop spaces of symplectic manifolds. A number of recent advances show that there is yet another manifestation of symplectic rigidity, taking place in function spaces associated to a symplectic manifold. These spaces exhibit unexpected properties and interesting structures, giving rise to an alternative intuition and new tools in symplectic topology, and providing a motivation to study the *function theory on symplectic manifolds*, which forms the subject of the present book.

Recall that a symplectic structure on a $2n$-dimensional manifold M is given by a closed differential 2-form ω which in appropriate local coordinates is given by $\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$. The Poisson bracket of a pair of smooth compactly supported functions F, G on M is a canonical operation given by

$$\{F, G\} = \sum_j \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

The Poisson bracket, which is one of our main characters, plays a fundamental role in symplectic geometry and its applications. For instance, it governs Hamiltonian mechanics. The symplectic manifold M serves as the phase space of a mechanical system. The evolution (or Hamiltonian flow) $h_t\colon M \to M$ of the system is determined by its time-dependent energy $H_t \in C^\infty(M)$. Hamilton's famous equation describing the motion of the system is given, in the Heisenberg picture, by $\dot{F}_t = \{F_t, H_t\}$, where $F_t = F \circ h_t$ stands for the time evolution of an observable function F on M under the Hamiltonian flow h_t. The diffeomorphisms h_t coming from all possible energies H_t form a group $\mathrm{Ham}(M, \omega)$, called the group of Hamiltonian diffeomorphisms. For closed simply connected manifolds this group is just the identity component of the symplectomorphism group. The group Ham can be considered as an infinite-dimensional Lie group. The function space $C^\infty(M)$ is, roughly speaking, the Lie algebra of this group, and the Poisson bracket is its Lie bracket.

The structure of the function theory we are going to develop can be illustrated with the help of the following picture. Fix your favorite $t > 0$

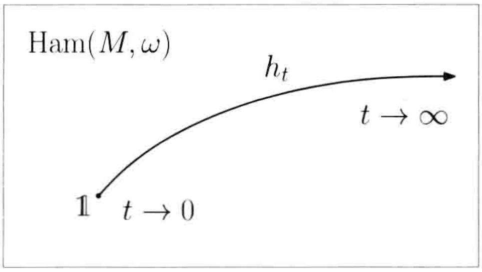

FIGURE 0.1. Two opposite regimes

and consider the natural mapping $C^\infty(M) \to \mathrm{Ham}(M)$ which takes a (time-independent) function H to the time-t map h_t of the corresponding Hamiltonian flow. In principle, this mapping enables one to translate information about Hamiltonian diffeomorphisms (which nowadays is quite a developed subject, see Chapter 4) into the language of function spaces. This naive plan works successfully in two opposite regimes, infinitesimal (when $t \to 0$) and asymptotic (when $t \to \infty$) (see Figure 0.1).

Working in the infinitesimal regime, one arrives at a surprising phenomenon of *C^0-robustness of the Poisson bracket.* Observe that the expression for the Poisson bracket involves the first derivatives of the functions F and G. Nevertheless, the functional $\Phi(F, G) := \|\{F, G\}\|$, where $\|\cdot\|$ stands for the uniform norm of a function, exhibits robustness with respect to C^0-perturbations. In particular, as we shall show in Chapter 2, Φ is lower semi-continuous in the uniform norm. Even though this result sounds analytical in nature, it turns out to be closely related to a remarkable bi-invariant geometry on the group $\mathrm{Ham}(M, \omega)$ discovered by Hofer in 1990. We shall discuss various facets of C^0-robustness of the Poisson bracket. One of them is the *Poisson bracket invariant* of a quadruple of subsets of a symplectic manifold discussed in Chapter 7. Its definition is based on an elementary looking variational problem involving the functional Φ, while its study involves a variety of methods of "hard" symplectic topology. Another facet is symplectic approximation theory, discussed in Chapter 8. Its basic objective is to find an optimal uniform approximation of a given pair of functions by a pair of (almost) Poisson commuting functions.

The asymptotic regime gives rise to the theory of *symplectic quasi-states* presented in Chapter 5. A symplectic quasi-state is a monotone functional $\zeta\colon C^\infty(M) \to \mathbb{R}$ with $\zeta(1) = 1$ which is linear on every Poisson-commutative subalgebra, but not necessarily on the whole function space. The origins of this notion go back to foundations of quantum mechanics and Aarnes' theory of topological quasi-states, an interesting branch of abstract functional analysis. In our context, nonlinear quasi-states on higher-dimensional manifolds are provided by Floer theory, the cornerstone of modern symplectic topology. Interestingly enough, symplectic quasi-states are closely related to *quasi-morphisms* on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$.

Roughly speaking, a quasi-morphism on a group is "a homomorphism up to a bounded error." This group-theoretical notion coming from bounded cohomology has been intensively studied in the past decade due to its various applications to geometry and dynamics. We discuss it in Chapter 3. A recent survey of quasi-states and quasi-morphisms in symplectic topology can be found in Entov's ICM-2014 talk [57].

Quasi-states serve as a useful tool for a number of problems in symplectic topology such as symplectic intersections, Hofer's geometry on groups of Hamiltonian diffeomorphisms, and Lagrangian knots. These applications are presented in Chapter 6. In addition, quasi-states provide yet another insight into robustness of the Poisson brackets, see Section 4.6.

Besides applications to some mainstream problems in symplectic topology, function theory on symplectic manifolds opens up a prospect of using "hard" symplectic methods in quantum mechanics. Mathematical quantization and, mostnotably, the quantum-classical correspondence principle provide a tool which enables one to translate basic notions of classical mechanics into quantum language. In general, a meaningful translation of symplectic rigidity phenomena involving subsets and diffeomorphisms faces serious analytical and conceptual difficulties. However, such a translation becomes possible if one shifts the focus from subsets and morphisms of manifolds to function spaces. We present some first steps in this direction in Chapter 9.

The book is a fusion of a research monograph on function theory on symplectic manifolds and an introductory survey of symplectic topology. On the introductory side, the first chapter discusses some basic symplectic constructions and fundamental phenomena, including the Eliashberg–Gromov C^0-rigidity theorem, Arnold's symplectic fixed point conjecture, and Hofer's metric, while in the last three chapters the reader will find an informal crash course on Floer theory. Even though our intention was to make the book as self-contained as possible, the reader is encouraged to consult earlier symplectic literature, such as the classical monographs [107, 108] by McDuff and Salamon. We also refer the reader to the manuscript by Oh [121] on Floer theory. The reader is assumed to have familiarity with basic differential and algebraic topology.

Most of the results presented in the book are based on a number of joint papers by L.P. with Michael Entov. L.P. expresses his gratitude to Michael for long years of pleasant collaboration. Furthermore, some central results of the book are joint with Lev Buhovsky (Poisson bracket invariants and symplectic approximation), Yakov Eliashberg (Lagrangian knots), and Frol Zapolsky (Poisson bracket inequality and rigidity of partitions of unity). L.P. cordially thanks all of them.

Parts of the material have been taught by L.P. in graduate courses at University of Chicago and Tel Aviv University, in a lecture series at UCLA, and (with the assistance of D.R.) in a mini-course at University of Melbourne. We thank these institutions for such an invaluable opportunity. We are indebted to Strom Borman, Lev Buhovsky, François Charette, Adi Dickstein, Michael Entov, Morimichi Kawasaki, Asaf Kislev, Karina Samvelyan, Egor Schelukhin, and Frol Zapolsky for very useful comments on the manuscript, as well as to Dorit Aharonov, Peter Albers, Paul Biran, Paul Busch, Danny Calegari, Yakov Eliashberg, Helmut Hofer, Vincente Humilière, Gil Kalai, Vadim Kaloshin, Yael Karshon, Guy Kindler, Dusa McDuff, Yong-Geun Oh, Yaron Ostrover, Iosif Polterovich, Victor Polterovich, Dietmar Salamon, Felix Schlenk, Paul Seidel, Sobhan Seyfaddini, Ivan Smith, Misha Sodin, Shmuel Weinberger, Amie Wilkinson, and Jinxin Xue for illuminating discussions on various topics of the book. Additionally, we thank the anonymous referees for very helpful critical remarks.

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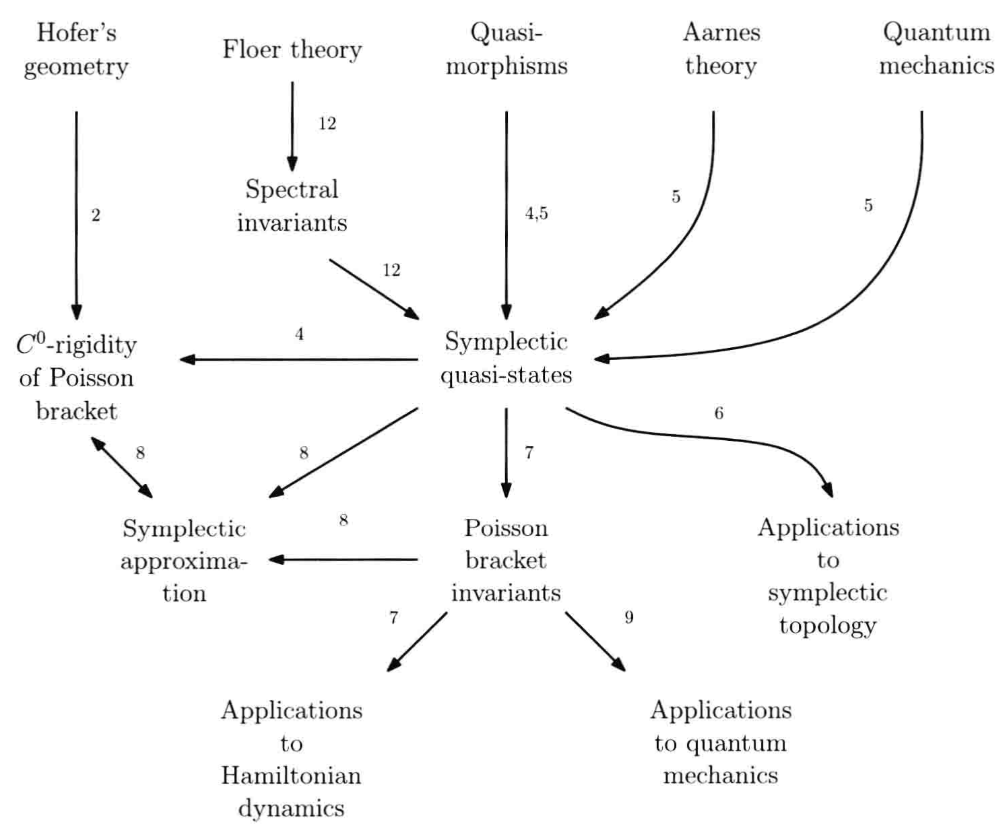

FIGURE 0.2. Subject road map. Numbers next to arrows indicate relevant chapters.

Contents

CHAPTER 1

Three Wonders of Symplectic Geometry

This chapter is a mixture of a rapid introduction to symplectic topology and a review of some of its landmark achievements. We mention three wonders of symplectic topology, the first being the Eliashberg–Gromov C^0-rigidity theorem [52, 77], which is the first of several manifestations of symplectic rigidity we will encounter in this book. The second is Arnold's conjecture [4, 6] about symplectic fixed points, which served as a driving force for many modern developments in symplectic topology, among them Floer theory, to which we will return towards the end of the book. The third wonder is Hofer's geometry on the group of Hamiltonian diffeomorphisms [85, 95, 131]. Lastly, we end the chapter with several examples of symplectic manifolds and a brief discussion of J-holomorphic curves.

1.1. First wonder: C^0-rigidity

Let M^{2n} be a smooth connected $2n$-dimensional manifold without boundary, and let ω be a closed 2-form on M which satisfies the following condition:

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \neq 0. \tag{1.1}$$

Then ω is called a *symplectic form*, and (M, ω) a *symplectic manifold.*

Note that ω^n is a top degree form, and hence by (1.1) a volume form, so in particular every symplectic manifold is orientable. The simplest examples include orientable surfaces equipped with area forms and their products. Here the product of two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is given by $(M_1 \times M_2, \omega_1 \oplus \omega_2)$. More sophisticated examples will be given at the end of this chapter.

A diffeomorphism f of a symplectic manifold (M, ω) is called a *symplectomorphism* if $f^*\omega = \omega$. The symplectomorphisms of (M, ω) form a group with respect to composition. We denote by $\mathrm{Symp}(M, \omega)$ the subgroup of all symplectomorphisms f with compact support: $fx = x$ for all points x outside a compact subset. By the C^k-topology, for $0 \le k \le \infty$, on $\mathrm{Symp}(M, \omega)$, and more generally, on the set of all diffeomorphisms of M, we mean the strong Whitney C^k-topology (see [84, Chapter 2]).

Note that symplectomorphisms are defined via their first derivatives, and hence the group $\mathrm{Symp}(M, \omega)$ is by its definition C^1-closed in the group $\mathrm{Diff}(M)$ of all compactly supported diffeomorphisms of M. However, for the same reason, a priori it is not obvious whether it is also C^0-closed.

A natural way to show that a certain class of transformations preserving a given tensor field on a manifold is C^0-closed is to characterize it by the conservation of a "C^0-robust" geometric quantity. Let us illustrate this by the following two examples:

• Let (M, g) be a closed Riemannian manifold. The group $\mathrm{Isom}(M, g)$ of all Riemannian isometries of (M, g) can be characterized by the preservation of the Riemannian distance $d(x, y)$ on M. If a sequence of diffeomorphisms f_k C^0-converges to a diffeomorphism f, we have that $d(f_k x, f_k y) \to d(fx, fy)$ for all $x, y \in M$. Thus if all f_k's are isometries, then $d(fx, fy) = d(x, y)$, and therefore f is also an isometry. We conclude that $\mathrm{Isom}(M, g)$ is C^0-closed in $\mathrm{Diff}(M)$.

• Let (M, σ) be an oriented manifold equipped with a volume form σ. The group $\mathrm{Diff}(M, \sigma)$ of all compactly supported σ-preserving diffeomorphisms of M can be characterized by the preservation of the volume $\int_U \sigma$ of open subsets $U \subset M$. It is easy to conclude from this that $\mathrm{Diff}(M, \sigma)$ is C^0-closed in $\mathrm{Diff}(M)$.

Even though no obvious candidate for such a C^0-robust quantity exists in the case of symplectomorphisms, the above phenomenon persists for symplectic manifolds [52; 77, Section 3.4.4]:

Theorem 1.1.1 (Eliashberg–Gromov rigidity theorem). *Let (M, ω) be a symplectic manifold. Then $\mathrm{Symp}(M, \omega)$ is C^0-closed in the group of all smooth compactly supported diffeomorphisms of M.*

We shall prove this result in Section 2.2 below by using methods of function theory on symplectic manifolds.

1.2. Second wonder: Arnold's conjecture

1.2.1. Mathematical model of classical mechanics. Before studying the properties of symplectic maps, a natural question to ask is whether such maps exist at all. For instance, a generic Riemannian metric on a manifold of dimension ≥ 2 admits no isometries except the identity map (see, e.g., [152, Proposition 1]). It turns out that in symplectic geometry the situation is quite different: an infinite-dimensional group of symplectomorphisms naturally arises within the mathematical model of classical mechanics. To describe this, we first need to discuss some linear algebra.

Let E^{2n} be a real vector space, equipped with an antisymmetric bilinear form $\omega \colon E \times E \to \mathbb{R}$. Define the map

$$I_\omega \colon E \to E^*, \quad \xi \mapsto i_\xi \omega = \omega(\xi, \cdot).$$

Exercise 1.2.1. Prove that the following conditions are equivalent:

(1) $\omega^n \neq 0$.
(2) I_ω is an isomorphism.

If the above equivalent conditions hold, E is called a *symplectic vector space*. For a more detailed account of symplectic vector spaces, we refer the reader to Chapter 4 in [107].

The basic example of a symplectic vector space is $\mathbb{R}^{2n}$, with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, equipped with the symplectic form $\omega_0 = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$, which we sometimes abbreviate to $\mathrm{d}p \wedge \mathrm{d}q$. According to the classical Darboux theorem, it provides a local model for the symplectic structure on an arbitrary symplectic manifold:

Theorem 1.2.2 (Darboux). *Let (M^{2n}, ω) be a symplectic manifold, and let $x \in M$. There exist local coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ near x such that in these coordinates we have $\omega = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$.*

Thus, symplectic manifolds have no local invariants, in contrast with Riemannian manifolds, which can be locally distinguished by their curvature tensor. We refer to these local coordinates as *a Darboux chart.* For proofs, see [107, Section 3.1] or [7, Section 43 B].

In classical mechanics, a symplectic manifold (M, ω) plays the role of the phase space of a system. A mechanical system is described by its Hamiltonian, or energy function, $H\colon M \times \mathcal{I} \to \mathbb{R}$. Here $\mathcal{I} \subset \mathbb{R}$ is a time interval which is usually assumed to contain 0. According to a basic principle of classical mechanics, the energy determines the time evolution of the system, in the following way.

We denote $H_t(x) := H(x,t)$. Define[1] the *symplectic gradient*, or Hamiltonian vector field, of H by

$$\operatorname{sgrad} H_t = -I_\omega^{-1}(\mathrm{d}H_t),$$

where $I_\omega\colon TM \to T^*M$ is the bundle isomorphism obtained by applying the isomorphism introduced in Exercise 1.2.1 fiber-wise. That is, for any vector field η on M,

$$\omega(\eta, \operatorname{sgrad} H_t) = \mathrm{d}H_t(\eta).$$

The *Hamilton equation* is the following ordinary differential equation on M:

$$\dot{x}(t) = \operatorname{sgrad} H_t\big(x(t)\big). \tag{1.2}$$

It gives rise to a one-parameter family of diffeomorphisms $\phi_H^t\colon M \to M$, defined by $\phi_H^t(x_0) = x(t)$, where $x(\cdot)$ is the unique solution of (1.2) with initial condition $x(0) = x_0$. (On noncompact manifolds, some extra assumptions on the behavior of H_t at infinity are required in order to guarantee that the solution $x(t)$ exists for all $t \in \mathcal{I}$.) The family $\{\phi_H^t\}$ is called the *Hamiltonian flow* of H, and each diffeomorphism in the family is called a *Hamiltonian diffeomorphism.*

Proposition 1.2.3. *The flow ϕ_H^t preserves ω.*

---

[1]Different authors may use different signs in the definitions of certain notions playing an important role in this book. This includes, in particular, Hamiltonian vector fields and Poisson brackets.

PROOF. The proof is a straightforward computation using Cartan's formula (we use the notation $\mathcal{L}_X$ for the Lie derivative with respect to the vector field X):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\phi_H^t)^*\omega = (\phi_H^t)^*\mathcal{L}_{\mathrm{sgrad}\,H_t}\omega &= (\phi_H^t)^*(\mathrm{d}i_{\mathrm{sgrad}\,H_t}\omega + i_{\mathrm{sgrad}\,H_t}\underbrace{\mathrm{d}\omega}_{=0}) \\ &= (\phi_H^t)^*(-\,\mathrm{d}^2 H_t) = 0.\end{aligned}$$

Hence, $(\phi_H^t)^*\omega = (\phi_H^0)^*\omega = \omega$ for all $t \in \mathcal{I}$. □

Therefore, any Hamiltonian diffeomorphism is a symplectomorphism of (M, ω).

Example 1.2.4. Consider $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \mathrm{d}p \wedge \mathrm{d}q$. We interpret q as the position vector, and p as the momentum. In this case the Hamilton equation (1.2) takes the form

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}; \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}. \end{cases}$$

Consider the particular case of the motion of a particle with mass m in $\mathbb{R}^n(q)$, in the presence of a potential $U(q)$ which depends only on the position. The force in this case is $F = -\partial U/\partial q$. The velocity of the particle is $\dot{q}$ and its momentum is defined by $p = m\dot{q}$. The Hamiltonian of the particle is its total energy $H(p,q) = K(p) + U(q)$, where

$$K = \frac{1}{2}m\dot{q}^2 = \frac{p^2}{2m}$$

is the *kinetic energy*. Thus,

$$H(p,q) = \frac{p^2}{2m} + U(q).$$

Hence, Hamilton's equations are

$$\begin{cases} \dot{q} = \dfrac{p}{m}; \\ \dot{p} = -\dfrac{\partial U}{\partial q} = F. \end{cases}$$

Combining these two equations, we obtain *Newton's second law*: $m\ddot{q} = F$. The passage from solving Newton's equation, a second order ODE in the configuration space $\mathbb{R}^n$, to Hamilton's equation, a first order ODE in the phase space $\mathbb{R}^{2n}$, brings classical mechanics into the framework of the theory of dynamical systems.

One of the first important results in classical mechanics was Liouville's theorem, which states that the time evolution of a mechanical system under Hamilton's equation preserves volume in the phase space. Since the natural