THE SELECTED WORKS OF RODERICK S. C. WONG

VOLUME 2

EDITORS

DAN DAI • HUI-HUI DAI

TONG YANG • DING-XUAN ZHOU

World Scientific

The Selected Works of Roderick S. C. Wong

Volume 2

Editors

Dan Dai

Hui-Hui Dai

Tong Yang

Ding-Xuan Zhou

City University of Hong Kong, Hong Kong

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Wong, Roderick, 1944–
[Works. Selections]
The selected works of Roderick S.C. Wong / edited by Dan Dai (City University of Hong Kong, Hong Kong), Hui-Hui Dai (City University of Hong Kong, Hong Kong), Tong Yang (City University of Hong Kong, Hong Kong), Ding-Xuan Zhou (City University of Hong Kong, Hong Kong).
3 volumes cm
Includes bibliographical references.
ISBN 978-9814656047 (set : hardcover : alk. paper) -- ISBN 9814656046 (set : hardcover : alk. paper) -- ISBN 978-9814656078 (vol. 1 : hardcover : alk. paper) -- ISBN 9814656070 (vol. 1 : hardcover : alk. paper) -- ISBN 978-9814656085 (vol. 2 : hardcover : alk. paper) -- ISBN 9814656089 (v. 2 : hardcover : alk. paper) -- ISBN 978-9814656092 (vol. 3 : hardcover : alk. paper) -- ISBN 9814656097 (v. 3 : hardcover : alk. paper)
1. Differential equations--Asymptotic theory. I. Dai, Dan, 1981– II. Title.
QA297.W66 2015
515'.35--dc23

2015003149

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore

The Selected Works of
Roderick S. C. Wong

Volume 2

Contents

Volume 3

Asymptotic Behavior of the Pollaczek Polynomials and Their Zeros

By Bo Rui and R. Wong

In 1954, A. Novikoff studied the asymptotic behavior of the Pollaczek polynomials $P_n(x; a, b)$ when $x = \cos(t/\sqrt{n})$, where $t > 0$ is *fixed*. He divided the positive t-axis into two regions, $0 < t < (a+b)^{1/2}$ and $t > (a+b)^{1/2}$, and derived an asymptotic formula in each of the two regions. Furthermore, he found an asymptotic formula for the zeros of these polynomials. Recently M. E. H. Ismail (1994) reconsidered this problem in an attempt to prove a conjecture of R. A. Askey and obtained a two-term expansion for these zeros. Here we derive an infinite asymptotic expansion for $P_n(\cos(t/\sqrt{n}); a, b)$, which holds uniformly for $0 < \varepsilon \le t \le M < \infty$, and show that Ismail's result is incorrect.

1. Introduction

In 1949 F. Pollaczek introduced a remarkable generalization of the Legendre polynomials. These polynomials $P_n(x; a, b)$ show in many respects a singular behavior; see [14, p. 393–396]. They are most easily defined by the generating function

$$\begin{aligned} f(x, w) = f(\cos\theta, w) &= \sum_{n=0}^{\infty} P_n(x; a, b) w^n \\ &= (1 - we^{i\theta})^{-1/2+ih(\theta)}(1 - we^{-i\theta})^{-1/2-ih(\theta)}, \end{aligned} \tag{1.1}$$

Address for correspondence: Professor R. Wong, Department of Mathematics, City University of Hong Kong, Tat Chee Ave., Kowloon, Hong Kong.

Published by Blackwell Publishers, 238 Main Street, Cambridge, MA 02142, USA, and 108 Cowley Road, Oxford, OX4 1JF, UK.

where

$$h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}, \qquad a > \pm b. \tag{1.2}$$

They reduce to the Legendre polynomial in the limiting case $a = b = 0$.

In 1954, A. Novikoff [12] wrote a well-known thesis on the asymptotics of these polynomials and their zeros. More specifically, he investigated the asymptotic behavior of $P_n(x; a, b)$ as $n \to \infty$, where $x = \cos(t/\sqrt{n})$ and $t > 0$ is *fixed*. He divided the positive t-axis into two regions $0 < t < (a+b)^{1/2}$ and $t > (a+b)^{1/2}$, and derived an asymptotic formula in each of the two regions. Furthermore, if the zeros of these polynomials are denoted by $\cos\theta$, where $0 < \theta_{1n} < \cdots < \theta_{nn} < \pi$, then he showed that for any fixed ν

$$\lim_{n\to\infty} n^{1/2}\theta_{\nu n} = (a+b)^{1/2}. \tag{1.3}$$

More recently R. A. Askey conjectured that the next term in the asymptotic expansion of $\theta_{\nu n}$ will involve zeros of a certain transcendental function. In an attempt to prove this conjecture, M. E. H. Ismail [10] derived a two-term expansion for $\theta_{\nu n}$, whose second term involves a zero of the entire function

$$F(\xi) = \int_0^1 (1-v^2)^{-1/2} e^{(a+b)v^2} \cos(\xi v)\, dv.$$

A drawback of Ismail's result is that $F(\xi)$ is not one of the familiar special functions of mathematical physics. Consequently, not much is known about the zeros of this function.

The purpose of this paper is to present an infinite asymptotic expansion for $P_n(\cos(t/\sqrt{n}); a, b)$, involving the Airy function and its derivative, which holds uniformly for $0 < \varepsilon \le t \le M < \infty$. Moreover, we show that $\theta_{\nu n}$ has the two-term asymptotic expansion

$$\theta_{\nu n} = \sqrt{\frac{a+b}{n}} + \frac{(a+b)^{1/6}(-a_\nu)}{2n^{5/6}} + O\left(\frac{1}{n^{7/6}}\right), \tag{1.4}$$

where a_ν is the νth negative zero of the Airy function $\mathrm{Ai}(\cdot)$. This result contradicts that of Ismail given in [10].

Our approach here is in spirit similar to that used in the derivations of the uniform asymptotic expansions of Laguerre polynomials [7] and Charlier polynomials [3]. However, the details of analysis in this paper, and especially in the discussion of the transformation (3.10) in Section 3, are quite different, and in fact are more difficult than those in the two previous papers.

2. Novikoff's results

From (1.1), one has by the Cauchy integral formula

$$P_n(x; a, b) = \frac{1}{2\pi i} \int_C \frac{f(\cos\theta, z)}{z^{n+1}} dz, \tag{2.1}$$

where C is a positively oriented simple closed curve surrounding $z = 0$ and not containing the branch points $z = e^{\pm i\theta}$. In view of the reflection formula [12, p. 7]

$$P_n(x; a, b) = (-1)^n P_n(-x; a, -b), \tag{2.2}$$

one may also assume without loss of generality that $0 < \theta < \pi/2$. In [12], Novikoff took C to be the contour C_1 shown in Figure 1. This contour consists of a large circle of radius $R \gg 1$, two small circles of radius $\delta \ll 1$, and two doubly traversed straight line segments, one vertical and one horizontal. The large circle is oriented in the counterclockwise direction, and the two small circles are oriented in the clockwise direction, so that $z = 0$ lies inside the region bounded by C_1, and $z = e^{\pm i\theta}$ lie outside this region. It can be shown, as in [12], that the integral along the large circle $|z| = R$ and the two small

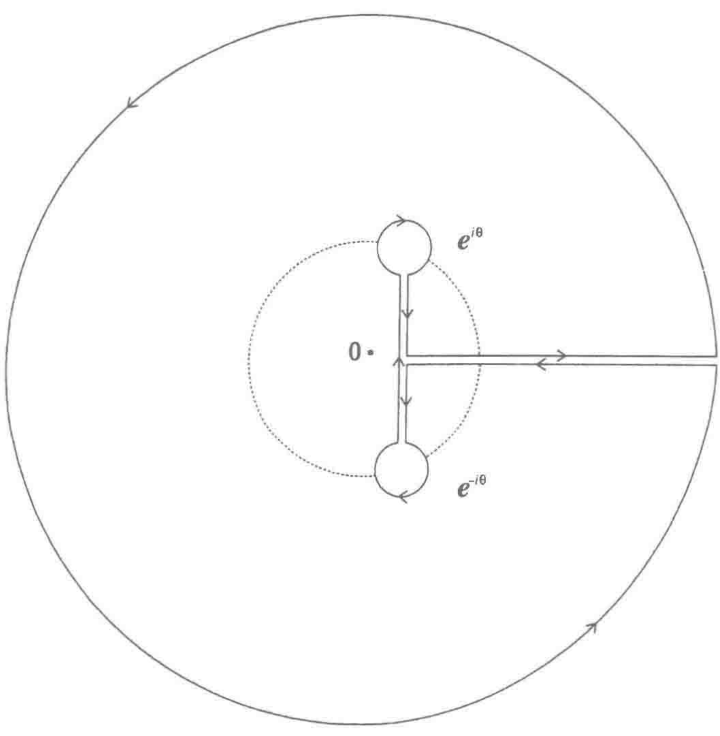

Figure 1. Contour C_1.

circles $|z - e^{\pm i\theta}| = \delta$ vanish as $\delta \to 0$ and $R \to \infty$. The contribution along the horizontal line segment $z = \cos\theta + \tau$, $0 \le \tau \le R$, is also zero, since $f(x, w)$ has the same value on this portion of the contour in either direction. Thus, the only contribution to the integral in (2.1), which does not vanish when $\delta \to 0$ and $R \to \infty$, comes from that portion of the integration path lying along the two sides of the cut $z = \cos\theta + i\alpha\sin\theta$, $-1 < \alpha < 1$, joining the two branch points $e^{i\theta}$ and $e^{-i\theta}$. Novikoff showed that along the left edge of the cut

$$f(x, z) = e^{(\pi - 2\theta)h(\theta)} \left(\frac{1+\alpha}{1-\alpha}\right)^{ih(\theta)} (\sin\theta)^{-1}(1-\alpha^2)^{-1/2}, \tag{2.3}$$

where $\arg((1+\alpha)/(1-\alpha)) = 0$ if $-1 < \alpha < 1$, and that this value is changed by a factor $-e^{-2\pi h(\theta)}$ along the right edge of cut. As a result, he obtained the integral representation

$$P_n(\cos\theta; a, b) = \frac{e^{-2\theta h(\theta)}\cosh(\pi h(\theta))}{\pi} \int_{-1}^{1} \frac{((1+\alpha)/(1-\alpha))^{ih(\theta)}}{(\cos\theta + i\alpha\sin\theta)^{n+1}} \cdot \frac{d\alpha}{(1-\alpha^2)^{1/2}}, \tag{2.4}$$

where $\log((1+\alpha)/(1-\alpha))$ has to be taken as real for $-1 < \alpha < 1$. This integral is an analogue of the Laplace integral representation for the Legendre polynomial, which it includes as a special case if $a = b = 0$.

To approximate $P_n(\cos\theta; a, b)$ when $\theta = t/\sqrt{n}$ and n is large, Novikoff noted that

$$h(\theta) = h\left(\frac{t}{\sqrt{n}}\right) = \frac{a+b}{2t}\sqrt{n} + O\left(\frac{1}{\sqrt{n}}\right) \tag{2.5}$$

and

$$-(n+1)\log\left(\cos\frac{t}{\sqrt{n}} + i\alpha\sin\frac{t}{\sqrt{n}}\right) = -i\alpha t\sqrt{n} + \frac{1-\alpha^2}{2}t^2 + O\left(\frac{1}{\sqrt{n}}\right). \tag{2.6}$$

Hence by putting

$$h(\alpha, t) \equiv \alpha t - \frac{a+b}{2t}\log\frac{1+\alpha}{1-\alpha}, \tag{2.7}$$

the last integral can be approximated by the simplified integral

$$J = \int_{-1}^{1} \exp\left\{-i\sqrt{n}h(\alpha, t) + \frac{1-\alpha^2}{2}t^2\right\}(1-\alpha^2)^{-1/2}d\alpha. \tag{2.8}$$

To the integral J, Novikoff applied the method of steepest descent [5, p. 65], which consists of finding the saddle points of $h(\alpha, t)$, i.e., the zeros of

$$\frac{dh}{d\alpha} = t - \frac{a+b}{t}\frac{1}{1-\alpha^2},$$

and deforming the linear path of integration into an appropriate one passing through the saddle points. The saddle points of h are given by

$$\alpha_{\pm} = \pm\left(1 - t^{-2}(a+b)\right)^{1/2}. \tag{2.9}$$

They are distinct as long as $t^2 \neq a+b$. If $t^2 > a+b$ then α_+ and α_- are real. If $t^2 < a+b$ then α_+ and α_- are conjugate imaginary. In both cases, α_+ and α_- are symmetrically located with respect to the origin $\alpha = 0$. By considering these two cases separately, Novikoff obtained the following results:

$$\begin{aligned} &P_n(\cos(t/\sqrt{n}); a, b) \\ &\quad = \frac{1}{2}\pi^{-1/2}(a+b-t^2)^{-1/4}\exp\left(-\frac{1}{2}(a+b)\right) \\ &\quad\quad \cdot n^{-1/4}\exp\left\{n^{1/2}\left(\frac{\pi}{2}\frac{a+b}{t} + \lambda(t)\right)\right\}\left(1 + O\left(\frac{1}{n^{1/4}}\right)\right), \end{aligned} \tag{2.10}$$

$$\lambda(t) = a_1 t - \frac{a+b}{t}\text{arc tan } a_1, \tag{2.11}$$

$$a_1 = t^{-1}(a+b-t^2)^{1/2}, \qquad 0 < t < (a+b)^{1/2}; \tag{2.12}$$

and

$$\begin{aligned} &P_n(\cos(t/\sqrt{n}); a, b) \\ &\quad = \pi^{-1/2}(t^2-a-b)^{-1/4}\exp\left(-\frac{1}{2}(a+b)\right) \\ &\quad\quad \cdot n^{-1/4}\exp\left\{n^{1/2}\left(\frac{\pi}{2}\frac{a+b}{t}\right)\right\}\cos\left(\frac{\pi}{4} - n^{1/2}\mu(t)\right) + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \tag{2.13}$$

$$\mu(t) = \alpha_1 t - \frac{a+b}{2t}\log\frac{1+\alpha_1}{1-\alpha_1}, \tag{2.14}$$

$$\alpha_1 = t^{-1}(t^2-a-b)^{1/2}, \qquad t > (a+b)^{1/2}. \tag{2.15}$$

3. Reduction to a canonical integral

For our purpose, we take C in (2.1) to be the contour C_2 shown in Figure 2, which consists of a large circle oriented in the counterclockwise direction, two small circles oriented in the clockwise direction, and two doubly traversed vertical infinite half-lines emitting from the branch points $z = e^{i\theta}$ and $z = e^{-i\theta}$.

As in [12], it can be shown that the integral along the large circle vanishes as $R \to \infty$. For the integral along the remaining portion of the contour, we make the change of variable $z = \cos\theta + i\alpha\sin\theta$, which is simply a linear

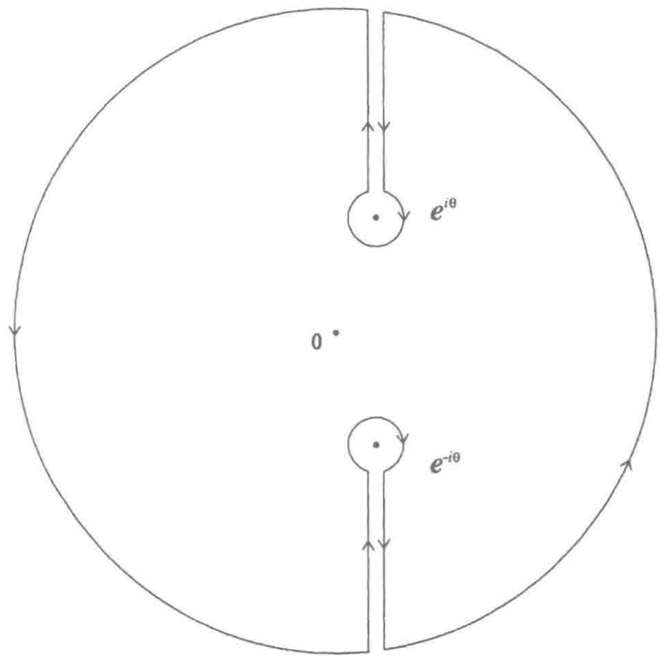

Figure 2. Contour C_2.

transformation consisting of a translation, a rotation, and a rescaling. The two vertical cuts in the z-plane are now mapped into the two horizontal half-lines along the real axis in the α-plane, one extending from $\alpha = 1$ to $\alpha = +\infty$ and the other from $\alpha = -1$ to $\alpha = -\infty$. In the cut α-plane, it can be shown that

$$f(\cos\theta, z) = e^{(\pi-2\theta)h(\theta)}\left(\frac{1+\alpha}{1-\alpha}\right)^{ih(\theta)} \frac{1}{(1-\alpha^2)^{1/2}\sin\theta}, \tag{3.1}$$

where $\arg((1+\alpha)/(1-\alpha)) = 0$ if $-1 < \alpha < 1$; compare with (2.3). Consequently, the integral (2.1) can be written as

$$P_n(\cos\theta; a, b) \\ = e^{(\pi-2\theta)h(\theta)}\frac{(-i)}{2\pi i}\left[\int_{\infty}^{(1^+)} + \int_{-\infty}^{(-1^+)}\right]\frac{[(1+\alpha)/(1-\alpha)]^{ih(\theta)}}{(\cos\theta + i\alpha\sin\theta)^{n+1}(1-\alpha^2)^{1/2}}d\alpha, \tag{3.2}$$

where the loop paths of integration are as indicated in Figure 3.

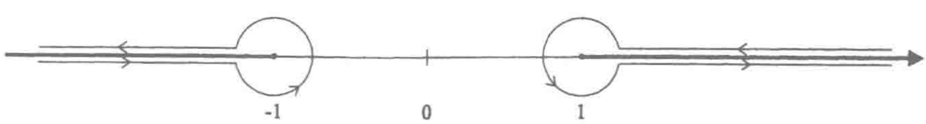

Figure 3. α = plane.