

THE SELECTED WORKS OF RODERICK S. C. WONG

VOLUME 2

EDITORS

DAN DAI • HUI-HUI DAI
TONG YANG • DING-XUAN ZHOU

THE SELECTED WORKS OF RODERICK S. C. WONG

VOLUME 2

EDITORS

DAN DAI

HUI-HUI DAI

TONG YANG

DING-XUAN ZHOU

City University of Hong Kong, Hong Kong



 World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by:

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Wong, Roderick, 1944–

[Works. Selections]

The selected works of Roderick S.C. Wong / edited by Dan Dai (City University of Hong Kong, Hong Kong), Hui-Hui Dai (City University of Hong Kong, Hong Kong), Tong Yang (City University of Hong Kong, Hong Kong), Ding-Xuan Zhou (City University of Hong Kong, Hong Kong).

3 volumes cm

Includes bibliographical references.

ISBN 978-9814656047 (set : hardcover : alk. paper) -- ISBN 9814656046 (set : hardcover : alk. paper) -- ISBN 978-9814656078 (vol. 1 : hardcover : alk. paper) -- ISBN 9814656070 (vol. 1 : hardcover : alk. paper) -- ISBN 978-9814656085 (vol. 2 : hardcover : alk. paper) -- ISBN 9814656089 (v. 2 : hardcover : alk. paper) -- ISBN 978-9814656092 (vol. 3 : hardcover : alk. paper) -- ISBN 9814656097 (v. 3 : hardcover : alk. paper)

1. Differential equations--Asymptotic theory. I. Dai, Dan, 1981– II. Title.

QA297.W66 2015

515'.35--dc23

2015003149

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Copyright © 2016 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

Printed in Singapore

THE SELECTED WORKS OF
RODERICK S. C. WONG

VOLUME 2

Contents

Preface		xi
Photos		xvii
Volume 1		
1.	The Asymptotic Behaviour of $\mu(z, \beta, \alpha)$, <i>Canad. J. Math.</i> , 21 (1969), 1013–1023.	1
2.	A Generalization of Watson’s Lemma, <i>Canad. J. Math.</i> , 24 (1972), 185–208.	12
3.	Linear Equations in Infinite Matrices, <i>Linear Algebra and Appl.</i> , 7 (1973), 53–62.	36
4.	Asymptotic Solutions of Linear Volterra Integral Equations with Singular Kernels, <i>Trans. Amer. Math. Soc.</i> , 189 (1974), 185–200.	46
5.	On Infinite Systems of Linear Differential Equations, <i>Canad. J. Math.</i> , 27 (1975), 691–703.	62
6.	Error Bounds for Asymptotic Expansions of Hankel Transforms, <i>SIAM J. Math. Anal.</i> , 7 (1976), 799–808.	75
7.	Explicit Error Terms for Asymptotic Expansions of Stieltjes Transforms, <i>J. Inst. Math. Appl.</i> , 22 (1978), 129–145.	85
8.	Explicit Error Terms for Asymptotic Expansions of Mellin Convolutions, <i>J. Math. Anal. Appl.</i> , 72 (1979), 740–756.	102
9.	Asymptotic Expansion of Multiple Fourier Transforms, <i>SIAM J. Math. Anal.</i> , 10 (1979), 1095–1104.	119

-
10. Exact Remainders for Asymptotic Expansions of Fractional Integrals, *J. Inst. Math. Appl.*, **23**(1979), 139–147. 129
 11. Asymptotic Expansion of the Hilbert Transform, *SIAM J. Math. Anal.*, **11**(1980), 92–99. 138
 12. Error Bounds for Asymptotic Expansions of Integrals, *SIAM Rev.*, **22**(1980), 401–435. 146
 13. Distributional Derivation of an Asymptotic Expansion, *Proc. Amer. Math. Soc.*, **80**(1980), 266–270. 181
 14. On a Method of Asymptotic Evaluation of Multiple Integrals, *Math. Comp.*, **37**(1981), 509–521. 186
 15. Asymptotic Expansion of the Lebesgue Constants Associated with Polynomial Interpolation, *Math. Comp.*, **39**(1982), 195–200. 199
 16. Quadrature Formulas for Oscillatory Integral Transforms, *Numer. Math.*, **39**(1982), 351–360. 205
 17. Generalized Mellin Convolutions and Their Asymptotic Expansions, *Canad. J. Math.*, **36**(1984), 924–960. 215
 18. A Uniform Asymptotic Expansion of the Jacobi Polynomials with Error Bounds, *Canad. J. Math.*, **37**(1985), 979–1007. 252
 19. Asymptotic Expansion of a Multiple Integral, *SIAM J. Math. Anal.*, **18**(1987), 1630–1637. 281
 20. Asymptotic Expansion of a Double Integral with a Curve of Stationary Points, *IMA J. Appl. Math.*, **38**(1987), 49–59. 289
 21. Szegő's Conjecture on Lebesgue Constants for Legendre Series, *Pacific J. Math.*, **135**(1988), 157–188. 300
 22. Uniform Asymptotic Expansions of Laguerre Polynomials, *SIAM J. Math. Anal.*, **19**(1988), 1232–1248. 332
 23. Transformation to Canonical Form for Uniform Asymptotic Expansions, *J. Math. Anal. Appl.*, **149**(1990), 210–219. 349

-
24. Multidimensional Stationary Phase Approximation: Boundary Stationary Point, *J. Comput. Appl. Math.*, **30**(1990), 213–225. 359
 25. Two-Dimensional Stationary Phase Approximation: Stationary Point at a Corner, *SIAM J. Math. Anal.*, **22**(1991), 500–523. 372
 26. Asymptotic Expansions for Second-Order Linear Difference Equations, *J. Comput. Appl. Math.*, **41**(1992), 65–94. 396
 27. Asymptotic Expansions for Second-Order Linear Difference Equations, II, *Stud. Appl. Math.*, **87**(1992), 289–324. 426
 28. Asymptotic Behaviour of the Fundamental Solution to $\partial u / \partial t = -(-\Delta)^m u$, *Proc. Roy. Soc. London Ser. A*, **441**(1993), 423–432. 462
 29. A Bernstein-Type Inequality for the Jacobi Polynomial, *Proc. Amer. Math. Soc.*, **121**(1994), 703–709. 472
 30. Error Bounds for Asymptotic Expansions of Laplace Convolutions, *SIAM J. Math. Anal.*, **25**(1994), 1537–1553. 479

Volume 2

31. Asymptotic Behavior of the Pollaczek Polynomials and Their Zeros, *Stud. Appl. Math.*, **96**(1996), 307–338. 497
32. Justification of the Stationary Phase Approximation in Time-Domain Asymptotics, *Proc. Roy. Soc. London Ser. A*, **453**(1997), 1019–1031. 529
33. Asymptotic Expansions of the Generalized Bessel Polynomials, *J. Comput. Appl. Math.*, **85**(1997), 87–112. 542
34. Uniform Asymptotic Expansions for Meixner Polynomials, *Constr. Approx.*, **14**(1998), 113–150. 568
35. “Best Possible” Upper and Lower Bounds for the Zeros of the Bessel Function $J_\nu(x)$, *Trans. Amer. Math. Soc.*, **351**(1999), 2833–2859. 606

36. Justification of a Perturbation Approximation of the Klein–Gordon Equation, *Stud. Appl. Math.*, **102**(1999), 375–417. 633
37. Smoothing of Stokes’s Discontinuity for the Generalized Bessel Function. II, *Proc. Roy. Soc. London Ser. A*, **455**(1999), 3065–3084. 676
38. Uniform Asymptotic Expansions of a Double Integral: Coalescence of Two Stationary Points, *Proc. Roy. Soc. London Ser. A*, **456**(2000), 407–431. 696
39. Uniform Asymptotic Formula for Orthogonal Polynomials with Exponential Weight, *SIAM J. Math. Anal.*, **31**(2000), 992–1029. 721
40. On the Asymptotics of the Meixner–Pollaczek Polynomials and Their Zeros, *Constr. Approx.*, **17**(2001), 59–90. 759
41. Gevrey Asymptotics and Stieltjes Transforms of Algebraically Decaying Functions, *Proc. Roy. Soc. London Ser. A*, **458**(2002), 625–644. 791
42. Exponential Asymptotics of the Mittag–Leffler Function, *Constr. Approx.*, **18**(2002), 355–385. 811
43. On the Ackerberg–O’Malley Resonance, *Stud. Appl. Math.*, **110**(2003), 157–179. 842
44. Asymptotic Expansions for Second-Order Linear Difference Equations with a Turning Point, *Numer. Math.*, **94**(2003), 147–194. 865
45. On a Two-Point Boundary-Value Problem with Spurious Solutions, *Stud. Appl. Math.*, **111**(2003), 377–408. 913
46. Shooting Method for Nonlinear Singularly Perturbed Boundary-Value Problems, *Stud. Appl. Math.*, **112**(2004), 161–200. 945

Volume 3

47. Asymptotic Expansion of the Krawtchouk Polynomials and Their Zeros, *Comput. Methods Funct. Theory*, **4**(1)(2004), 189–226. 985
48. On a Uniform Treatment of Darboux's Method, *Constr. Approx.*, **21**(2005), 225–255. 1023
49. Linear Difference Equations with Transition Points, *Math. Comp.*, **74**(2005), 629–653. 1054
50. Uniform Asymptotics for Jacobi Polynomials with Varying Large Negative Parameters — A Riemann–Hilbert Approach, *Trans. Amer. Math. Soc.*, **358**(2006), 2663–2694. 1079
51. Uniform Asymptotics of the Stieltjes–Wigert Polynomials via the Riemann–Hilbert Approach, *J. Math. Pures Appl.*, **85**(5)(2006), 698–718. 1111
52. A Singularly Perturbed Boundary-Value Problem Arising in Phase Transitions, *European J. Appl. Math.*, **17**(6)(2006), 705–733. 1132
53. On the Number of Solutions to Carrier's Problem, *Stud. Appl. Math.*, **120**(3)(2008), 213–245. 1161
54. Asymptotic Expansions for Riemann–Hilbert Problems, *Anal. Appl. (Singap.)*, **6**(2008), 269–298. 1194
55. On the Connection Formulas of the Third Painlevé Transcendent, *Discrete Contin. Dyn. Syst.*, **23**(2009), 541–560. 1224
56. Hyperasymptotic Expansions of the Modified Bessel Function of the Third Kind of Purely Imaginary Order, *Asymptot. Anal.*, **63**(2009), 101–123. 1244
57. Global Asymptotics for Polynomials Orthogonal with Exponential Quartic Weight, *Asymptot. Anal.*, **64**(2009), 125–154. 1267

58.	The Riemann–Hilbert Approach to Global Asymptotics of Discrete Orthogonal Polynomials with Infinite Nodes, <i>Anal. Appl. (Singap.)</i> , 8 (2010), 247–286.	1297
59.	Global Asymptotics of the Meixner Polynomials, <i>Asymptot. Anal.</i> , 75 (2011), 211–231.	1337
60.	Asymptotics of Orthogonal Polynomials via Recurrence Relations, <i>Anal. Appl. (Singap.)</i> , 10 (2)(2012), 215–235.	1358
61.	Uniform Asymptotic Expansions for the Discrete Chebyshev Polynomials, <i>Stud. Appl. Math.</i> , 128 (2012), 337–384.	1379
62.	Global Asymptotics of the Hahn Polynomials, <i>Anal. Appl. (Singap.)</i> , 11 (3)(2013), 1350018.	1427
63.	Global Asymptotics of Stieltjes–Wigert Polynomials, <i>Anal. Appl. (Singap.)</i> , 11 (5)(2013), 1350028.	1474
	List of Publications by Roderick S. C. Wong	1487
	Permissions	1501

Asymptotic Behavior of the Pollaczek Polynomials and Their Zeros

By Bo Rui and R. Wong

In 1954, A. Novikoff studied the asymptotic behavior of the Pollaczek polynomials $P_n(x; a, b)$ when $x = \cos(t/\sqrt{n})$, where $t > 0$ is fixed. He divided the positive t -axis into two regions, $0 < t < (a+b)^{1/2}$ and $t > (a+b)^{1/2}$, and derived an asymptotic formula in each of the two regions. Furthermore, he found an asymptotic formula for the zeros of these polynomials. Recently M. E. H. Ismail (1994) reconsidered this problem in an attempt to prove a conjecture of R. A. Askey and obtained a two-term expansion for these zeros. Here we derive an infinite asymptotic expansion for $P_n(\cos(t/\sqrt{n}); a, b)$, which holds uniformly for $0 < \varepsilon \leq t \leq M < \infty$, and show that Ismail's result is incorrect.

1. Introduction

In 1949 F. Pollaczek introduced a remarkable generalization of the Legendre polynomials. These polynomials $P_n(x; a, b)$ show in many respects a singular behavior; see [14, p. 393–396]. They are most easily defined by the generating function

$$\begin{aligned} f(x, w) = f(\cos \theta, w) &= \sum_{n=0}^{\infty} P_n(x; a, b) w^n \\ &= (1 - we^{i\theta})^{-1/2+ih(\theta)} (1 - we^{-i\theta})^{-1/2-ih(\theta)}, \end{aligned} \quad (1.1)$$

Address for correspondence: Professor R. Wong, Department of Mathematics, City University of Hong Kong, Tat Chee Ave., Kowloon, Hong Kong.

where

$$h(\theta) = \frac{a \cos \theta + b}{2 \sin \theta}, \quad a > \pm b. \quad (1.2)$$

They reduce to the Legendre polynomial in the limiting case $a = b = 0$.

In 1954, A. Novikoff [12] wrote a well-known thesis on the asymptotics of these polynomials and their zeros. More specifically, he investigated the asymptotic behavior of $P_n(x; a, b)$ as $n \rightarrow \infty$, where $x = \cos(t/\sqrt{n})$ and $t > 0$ is fixed. He divided the positive t -axis into two regions $0 < t < (a+b)^{1/2}$ and $t > (a+b)^{1/2}$, and derived an asymptotic formula in each of the two regions. Furthermore, if the zeros of these polynomials are denoted by $\cos \theta$, where $0 < \theta_{1n} < \dots < \theta_{nn} < \pi$, then he showed that for any fixed ν

$$\lim_{n \rightarrow \infty} n^{1/2} \theta_{\nu n} = (a+b)^{1/2}. \quad (1.3)$$

More recently R. A. Askey conjectured that the next term in the asymptotic expansion of $\theta_{\nu n}$ will involve zeros of a certain transcendental function. In an attempt to prove this conjecture, M. E. H. Ismail [10] derived a two-term expansion for $\theta_{\nu n}$, whose second term involves a zero of the entire function

$$F(\xi) = \int_0^1 (1-v^2)^{-1/2} e^{(a+b)v^2} \cos(\xi v) dv.$$

A drawback of Ismail's result is that $F(\xi)$ is not one of the familiar special functions of mathematical physics. Consequently, not much is known about the zeros of this function.

The purpose of this paper is to present an infinite asymptotic expansion for $P_n(\cos(t/\sqrt{n}); a, b)$, involving the Airy function and its derivative, which holds uniformly for $0 < \varepsilon \leq t \leq M < \infty$. Moreover, we show that $\theta_{\nu n}$ has the two-term asymptotic expansion

$$\theta_{\nu n} = \sqrt{\frac{a+b}{n}} + \frac{(a+b)^{1/6}(-a_\nu)}{2n^{5/6}} + O\left(\frac{1}{n^{7/6}}\right), \quad (1.4)$$

where a_ν is the ν th negative zero of the Airy function $\text{Ai}(\cdot)$. This result contradicts that of Ismail given in [10].

Our approach here is in spirit similar to that used in the derivations of the uniform asymptotic expansions of Laguerre polynomials [7] and Charlier polynomials [3]. However, the details of analysis in this paper, and especially in the discussion of the transformation (3.10) in Section 3, are quite different, and in fact are more difficult than those in the two previous papers.

2. Novikoff's results

From (1.1), one has by the Cauchy integral formula

$$P_n(x; a, b) = \frac{1}{2\pi i} \int_C \frac{f(\cos \theta, z)}{z^{n+1}} dz, \quad (2.1)$$

where C is a positively oriented simple closed curve surrounding $z = 0$ and not containing the branch points $z = e^{\pm i\theta}$. In view of the reflection formula [12, p. 7]

$$P_n(x; a, b) = (-1)^n P_n(-x; a, -b), \quad (2.2)$$

one may also assume without loss of generality that $0 < \theta < \pi/2$. In [12], Novikoff took C to be the contour C_1 shown in Figure 1. This contour consists of a large circle of radius $R \gg 1$, two small circles of radius $\delta \ll 1$, and two doubly traversed straight line segments, one vertical and one horizontal. The large circle is oriented in the counterclockwise direction, and the two small circles are oriented in the clockwise direction, so that $z = 0$ lies inside the region bounded by C_1 , and $z = e^{\pm i\theta}$ lie outside this region. It can be shown, as in [12], that the integral along the large circle $|z| = R$ and the two small

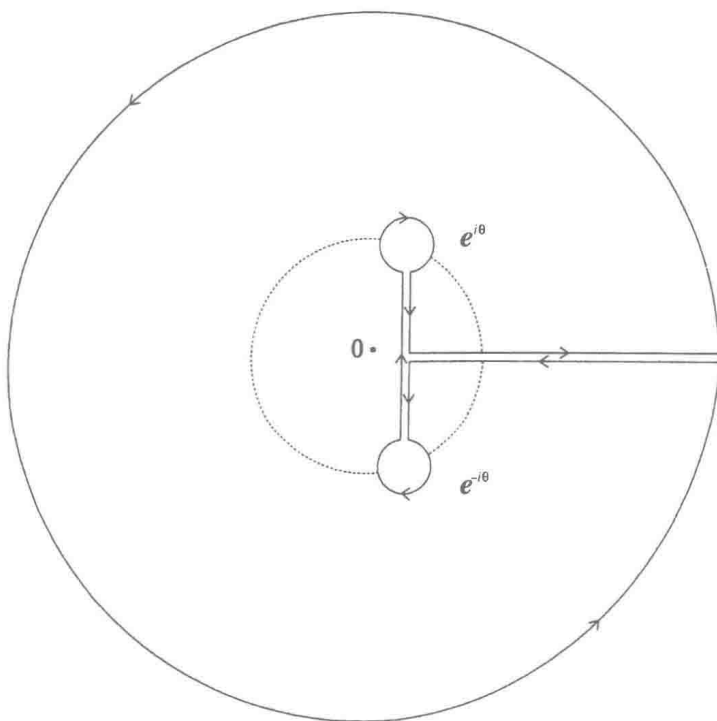


Figure 1. Contour C_1 .

circles $|z - e^{\pm i\theta}| = \delta$ vanish as $\delta \rightarrow 0$ and $R \rightarrow \infty$. The contribution along the horizontal line segment $z = \cos \theta + \tau$, $0 \leq \tau \leq R$, is also zero, since $f(x, w)$ has the same value on this portion of the contour in either direction. Thus, the only contribution to the integral in (2.1), which does not vanish when $\delta \rightarrow 0$ and $R \rightarrow \infty$, comes from that portion of the integration path lying along the two sides of the cut $z = \cos \theta + i\alpha \sin \theta$, $-1 < \alpha < 1$, joining the two branch points $e^{i\theta}$ and $e^{-i\theta}$. Novikoff showed that along the left edge of the cut

$$f(x, z) = e^{(\pi-2\theta)h(\theta)} \left(\frac{1+\alpha}{1-\alpha} \right)^{ih(\theta)} (\sin \theta)^{-1} (1-\alpha^2)^{-1/2}, \quad (2.3)$$

where $\arg((1+\alpha)/(1-\alpha)) = 0$ if $-1 < \alpha < 1$, and that this value is changed by a factor $-e^{-2\pi h(\theta)}$ along the right edge of cut. As a result, he obtained the integral representation

$$P_n(\cos \theta; a, b) = \frac{e^{-2\theta h(\theta)} \cosh(\pi h(\theta))}{\pi} \int_{-1}^1 \frac{((1+\alpha)/(1-\alpha))^{ih(\theta)}}{(\cos \theta + i\alpha \sin \theta)^{n+1}} \cdot \frac{d\alpha}{(1-\alpha^2)^{1/2}}, \quad (2.4)$$

where $\log((1+\alpha)/(1-\alpha))$ has to be taken as real for $-1 < \alpha < 1$. This integral is an analogue of the Laplace integral representation for the Legendre polynomial, which it includes as a special case if $a = b = 0$.

To approximate $P_n(\cos \theta; a, b)$ when $\theta = t/\sqrt{n}$ and n is large, Novikoff noted that

$$h(\theta) = h\left(\frac{t}{\sqrt{n}}\right) = \frac{a+b}{2t} \sqrt{n} + O\left(\frac{1}{\sqrt{n}}\right) \quad (2.5)$$

and

$$-(n+1) \log\left(\cos \frac{t}{\sqrt{n}} + i\alpha \sin \frac{t}{\sqrt{n}}\right) = -i\alpha t \sqrt{n} + \frac{1-\alpha^2}{2} t^2 + O\left(\frac{1}{\sqrt{n}}\right). \quad (2.6)$$

Hence by putting

$$h(\alpha, t) \equiv \alpha t - \frac{a+b}{2t} \log \frac{1+\alpha}{1-\alpha}, \quad (2.7)$$

the last integral can be approximated by the simplified integral

$$J = \int_{-1}^1 \exp\left\{-i\sqrt{n}h(\alpha, t) + \frac{1-\alpha^2}{2} t^2\right\} (1-\alpha^2)^{-1/2} d\alpha. \quad (2.8)$$

To the integral J , Novikoff applied the method of steepest descent [5, p. 65], which consists of finding the saddle points of $h(\alpha, t)$, i.e., the zeros of

$$\frac{dh}{d\alpha} = t - \frac{a+b}{t} \frac{1}{1-\alpha^2},$$

and deforming the linear path of integration into an appropriate one passing through the saddle points. The saddle points of h are given by

$$\alpha_{\pm} = \pm(1 - t^{-2}(a + b))^{1/2}. \quad (2.9)$$

They are distinct as long as $t^2 \neq a + b$. If $t^2 > a + b$ then α_+ and α_- are real. If $t^2 < a + b$ then α_+ and α_- are conjugate imaginary. In both cases, α_+ and α_- are symmetrically located with respect to the origin $\alpha = 0$. By considering these two cases separately, Novikoff obtained the following results:

$$\begin{aligned} P_n(\cos(t/\sqrt{n}); a, b) &= \frac{1}{2} \pi^{-1/2} (a + b - t^2)^{-1/4} \exp\left(-\frac{1}{2}(a + b)\right) \\ &\cdot n^{-1/4} \exp\left\{n^{1/2} \left(\frac{\pi}{2} \frac{a + b}{t} + \lambda(t)\right)\right\} \left(1 + O\left(\frac{1}{n^{1/4}}\right)\right), \end{aligned} \quad (2.10)$$

$$\lambda(t) = a_1 t - \frac{a + b}{t} \arctan a_1, \quad (2.11)$$

$$a_1 = t^{-1}(a + b - t^2)^{1/2}, \quad 0 < t < (a + b)^{1/2}; \quad (2.12)$$

and

$$\begin{aligned} P_n(\cos(t/\sqrt{n}); a, b) &= \pi^{-1/2} (t^2 - a - b)^{-1/4} \exp\left(-\frac{1}{2}(a + b)\right) \\ &\cdot n^{-1/4} \exp\left\{n^{1/2} \left(\frac{\pi}{2} \frac{a + b}{t}\right)\right\} \cos\left(\frac{\pi}{4} - n^{1/2} \mu(t)\right) + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \quad (2.13)$$

$$\mu(t) = \alpha_1 t - \frac{a + b}{2t} \log \frac{1 + \alpha_1}{1 - \alpha_1}, \quad (2.14)$$

$$\alpha_1 = t^{-1}(t^2 - a - b)^{1/2}, \quad t > (a + b)^{1/2}. \quad (2.15)$$

3. Reduction to a canonical integral

For our purpose, we take C in (2.1) to be the contour C_2 shown in Figure 2, which consists of a large circle oriented in the counterclockwise direction, two small circles oriented in the clockwise direction, and two doubly traversed vertical infinite half-lines emitting from the branch points $z = e^{i\theta}$ and $z = e^{-i\theta}$.

As in [12], it can be shown that the integral along the large circle vanishes as $R \rightarrow \infty$. For the integral along the remaining portion of the contour, we make the change of variable $z = \cos \theta + i\alpha \sin \theta$, which is simply a linear

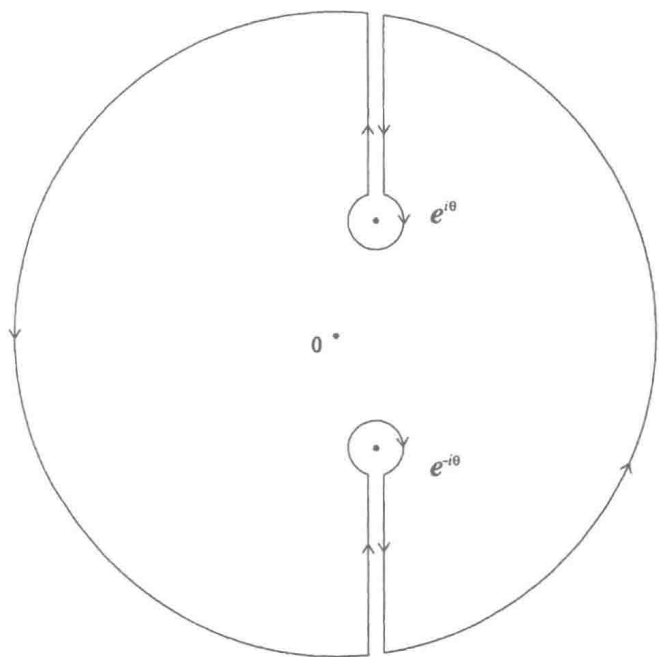


Figure 2. Contour C_2 .

transformation consisting of a translation, a rotation, and a rescaling. The two vertical cuts in the z -plane are now mapped into the two horizontal half-lines along the real axis in the α -plane, one extending from $\alpha = 1$ to $\alpha = +\infty$ and the other from $\alpha = -1$ to $\alpha = -\infty$. In the cut α -plane, it can be shown that

$$f(\cos \theta, z) = e^{(\pi-2\theta)h(\theta)} \left(\frac{1+\alpha}{1-\alpha} \right)^{ih(\theta)} \frac{1}{(1-\alpha^2)^{1/2} \sin \theta}, \tag{3.1}$$

where $\arg((1+\alpha)/(1-\alpha)) = 0$ if $-1 < \alpha < 1$; compare with (2.3). Consequently, the integral (2.1) can be written as

$$\begin{aligned} &P_n(\cos \theta; a, b) \\ &= e^{(\pi-2\theta)h(\theta)} \frac{(-i)}{2\pi i} \left[\int_{\infty}^{(1^+)} + \int_{(-1^+)}^{(-\infty)} \right] \frac{[(1+\alpha)/(1-\alpha)]^{ih(\theta)}}{(\cos \theta + i\alpha \sin \theta)^{n+1} (1-\alpha^2)^{1/2}} d\alpha, \end{aligned} \tag{3.2}$$

where the loop paths of integration are as indicated in Figure 3.

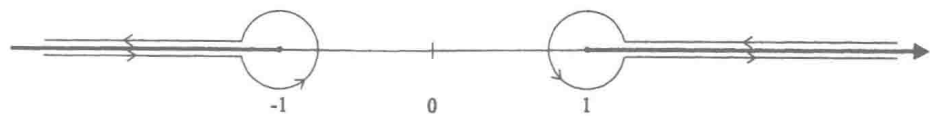


Figure 3. $\alpha =$ plane.