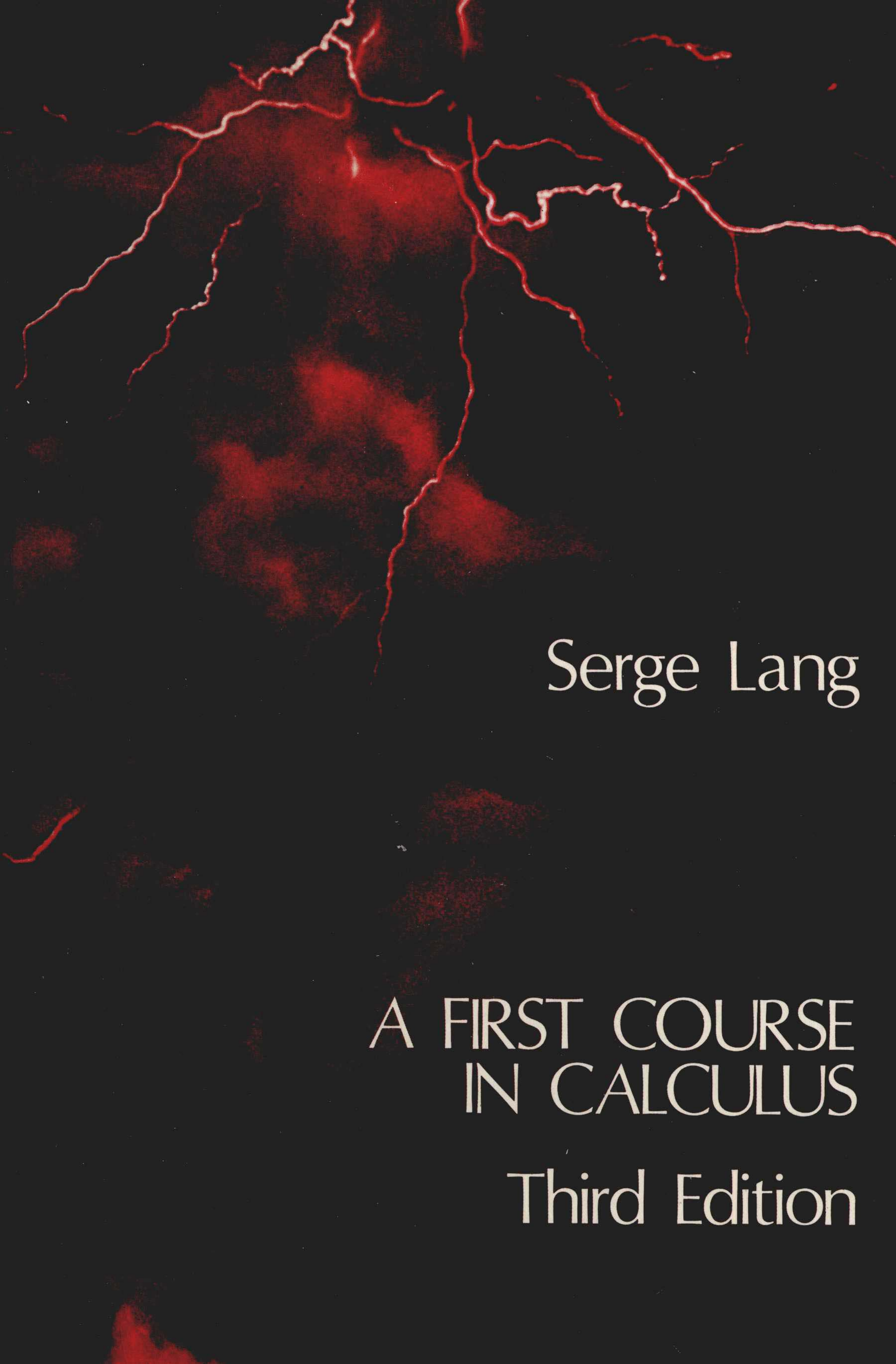

Serge Lang

# A FIRST COURSE IN CALCULUS

Third Edition

# A First Course in CALCULUS

**SERGE LANG**
Yale University

Third Edition

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# To the Third Edition

The text has been changed mostly by the addition of a large number of worked-out examples, and the rewriting of some sections.

Because many students from high schools have often been exposed to some calculus, one can cover the material more rapidly than used to be the case. Nevertheless, I have decided to preserve the first two chapters, for the convenience of those coming to the study of Calculus with shaky background.

It will also be frequently useful to cover during the first year those portions of the theory of differentiation in several variables which can be handled essentially by one-variable methods, combined with a minimum from the theory of vectors. Hence I have included in the present volume four chapters from the second course, *Calculus of Several Variables*, dealing with that part of the subject.

The increase in size of the book from its first edition is due to the factors mentioned above, and seems justified in view of the increased flexibility as a text.

SERGE LANG

# Foreword

The purpose of a first course in Calculus is to teach the student the basic notions of derivative and integral, and the basic techniques and applications which accompany them.

At present in the United States, the trend is to introduce Calculus in high schools, and I agree that the material covered in the present book should ultimately be the standard fare of the last two years of secondary schools.

Irrespective of when it is taught, I believe that the presentation remains more or less invariant. The very talented student, with an obvious aptitude for mathematics, will rapidly require a course in functions of one real variable, more or less as it is understood by professional mathematicians. This book is not primarily addressed to him (although I hope he will be able to acquire from it a good introduction at an early age).

I have not written this course in the style I would use for an advanced monograph, on sophisticated topics. One writes an advanced monograph for oneself, because one wants to give permanent form to one's vision of some beautiful part of mathematics, not otherwise accessible, somewhat in the manner of a composer setting down his symphony in musical notation.

This book is written for the student, to give him an immediate, and pleasant, access to the subject. I hope that I have struck a proper compromise between dwelling too much on special details, and not giving enough technical exercises, necessary to acquire the desired familiarity with the subject. In any case, certain routine habits of sophisticated mathematicians are unsuitable for a first course.

This does not mean that so-called rigor has to be abandoned. The logical development of the mathematics of this course from the most basic axioms proceeds through the following stages:

Set theory
Integers (whole numbers)
Rational numbers (fractions)
Numbers (i.e. real numbers)
Limits
Derivatives
and forward.

No one in his right mind suggests that one should begin a course with set theory. It happens that the most satisfactory place to jump into the subject is between limits and derivatives. In other words, any student is ready to accept as intuitively obvious the notions of numbers and limits and their basic properties. For some reason, it has become fashionable to hold that the best place to enter the subject is between numbers and limits. Experience shows that the students do *not* have the proper psychological background to accept this, and resist it tremendously.

In fact, it turns out that one can have the best of both ideas. The arguments which show how the properties of limits can be reduced to those of numbers form a self-contained whole. Logically, it belongs *before* the subject matter of our course. Nevertheless, we have inserted it as an appendix. If any student feels the need for it, he need but read it and visualize it as Chapter 0. In that case, everything that follows is as rigorous as any mathematician would wish it (so far as objects which receive an analytic definition are concerned). Not one word need be changed in any proof. I hope this takes care once and for all of possible controversies concerning so-called rigor.

Some objects receive a geometric definition, and there are applications to physical concepts. In that case, it is of course necessary to insert one step to bridge the physical notion and its mathematical counterpart. The major instances of this are the functions sine and cosine, and the area, as an integral.

For sine and cosine, we rely on the notions of plane geometry. If one accepts standard theorems concerning plane figures, then our proofs satisfy the above-mentioned standards. An appendix shows how one can give purely analytic definitions and proofs for the basic properties.

For the integral, we first give a geometric argument. We then show, using the usual Riemann sums, how this geometric argument has a perfect counterpart when we require the rules of the game to reduce all definitions and proofs to numbers. This should satisfy everybody. Furthermore, the theory of the integral is so presented that only its existence depends either on a geometric argument or a slightly involved theoretical investigation (upper and lower sums). According to the level of ability of a class, the teacher may therefore dose the theory according to his judgment.

It is not generally recognized that some of the major difficulties in teaching mathematics are analogous to those in teaching a foreign language. (The secondary schools are responsible for this. Proper training in the secondary schools could entirely eliminate this difficulty.) Consequently, I have made great efforts to carry the student verbally, so to say, in using proper mathematical language. It seems to me essential that students be required to write their mathematics papers in full and coherent sentences. A large portion of their difficulties with mathematics stems from their slapping down mathematical symbols and formulas isolated from a meaningful sentence and appropriate quantifiers. Papers should also be required to be neat and legible. They should not look as if a stoned fly had just crawled out of an inkwell. Insisting on reasonable standards of expression will result in drastic improvements of mathematical performance.

I believe that it is unsound to view "theory" as adversary to "computation". The present book treats both as complementary to each other. Almost always a theorem gives a tool for more efficient computations (e.g. Taylor's formula, for computing values of functions). Different classes will of course put different emphasis on them, omitting some proofs, but I have found that if no excessive pedantry is introduced, students are willing to understand the reasons for the truth of a result, i.e. its proof.

I have made no great innovations in the exposition of calculus. Since the subject was discovered some 300 years ago, it was out of the question. Rather, I have omitted some specialized topics which no longer belong in the curriculum. Stirling's formula is included only for reference, and can be skipped, or used to provide exercises. Taylor's formula is proved with the integral form of the remainder, which is then properly estimated. The proof with integration by parts is more natural than the other (differentiating some complicated expression pulled out of nowhere), and is the one which generalizes to the higher dimensional case. I have placed integration after differentiation, because otherwise one has no technique available to evaluate integrals. But on the whole, everything is fairly standard.

I have cut down the amount of analytic geometry to what is both necessary and sufficient for a general first course in this type of mathematics. For some applications, more is required, but these applications are fairly specialized. For instance, if one needs the special properties concerning the focus of a parabola in a course on optics, then that is the place to present them, not in a general course which is to serve mathematicians, physicists, chemists, biologists, and engineers, to mention but a few. I regard the tremendous emphasis on the analytic geometry of conics which has been the fashion for many years as an unfortunate historical accident. What is important is that the basic idea of representing a graph by a figure in the plane should be thoroughly understood, together with basic examples. The more abstruse properties of ellipses, parabolas, and hyperbolas should be skipped.

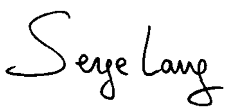
As for the question: Why write one more calculus book? I would answer: Because practically all existing ones are too long (500 to 600 pages) and one loses sight of the over-all ideas, sacrificed for the sake of topics which have hung on through habits, bad habits, I would say. I hope that the present arrangement of the chapters will give the reader a solid view of the subject.

To conclude, if I may be allowed another personal note here, I learned how to teach the present course from Artin, the year I wrote my Doctor's thesis. I could not have had a better introduction to the subject.

*New Haven, Connecticut*
*October 1972*

SERGE LANG

*My publishers, Addison-Wesley, have produced my books for these last ten years. I want it known how much I appreciate their extraordinary performance at all levels: general editorial advice, specific editing of the manuscripts, and essentially flawless typesetting and proof sheets. It is very gratifying to have found such a company to deal with.*



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# PART ONE

# REVIEW OF BASIC MATERIAL

If you are already at ease with the elementary properties of numbers and if you know about coordinates and the graphs of the standard equations (linear equations, parabolas, ellipses), then you should start immediately with Chapter III on derivatives.