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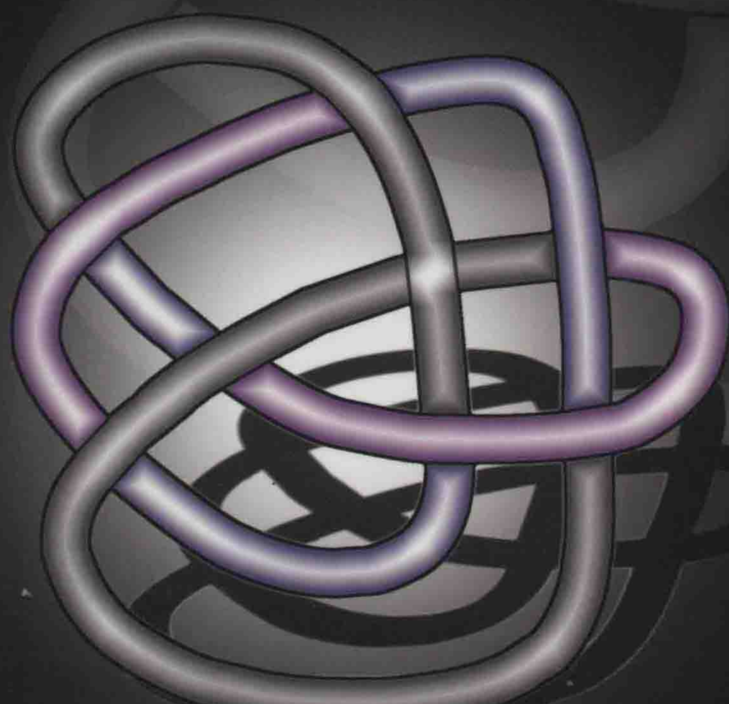
Quandles

An Introduction to the Algebra of Knots

Mohamed Elhamdadi
Sam Nelson



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An Introduction to the Algebra of Knots

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Quandles

An Introduction to the Algebra of Knots

Preface

Quandles and their kin (kei, racks, biquandles and biracks) are algebraic structures whose axioms encode the movements of knots in space in the same way that groups encode symmetry and orthogonal transformations encode rigid motion. Quandle theory thus brings together aspects of topology, abstract algebra and combinatorics in a way that is easily accessible using pictures and diagrams.

The term "quandle" was coined by David Joyce in his PhD dissertation, written in 1980 and published in 1982 [**Joy82**]. Previous work had been done as far back as 1942 by Mituhisa Takasaki [**Tak42**], who used the term "kei" for what Joyce would later call "involutory quandles". In the 1950s Conway and Wraith [**CW**] informally discussed a similar structure they called "wracks" from the phrase "wrack and ruin". At the same time Joyce was writing about quandles, Sergey V. Matveev [**Mat82**] was writing behind the iron curtain about the same algebraic structure, using the more descriptive term "distributive groupoids". Louis Kauffman [**Kau91**] used the term "crystals" for a form of the quandle structure. In the mid 1980s a generalized form of the quandle idea was independently discovered by Brieskorn [**Bri88**], who chose the descriptive term "automorphic sets".

In 1992 Roger Fenn and Colin Rourke [**FR92**] wrote a seminal work reintroducing the quandle idea and a generalization; they chose to use the Conway/Wraith term "wracks" while dropping the "w" to obtain the term "racks", canceling the "w" along with the writhe independence. In subsequent work [**FRS95**] they suggested a further generalization known as "biracks" with a special case known as "biquandles". Biquandles were explored in detail in 2002 by Louis Kauffman and David Radford [**KR03**], with later work by others [**CES04, FRS95, NV06**].

Fenn, Rourke and Sanderson introduced in [**FRS95**] a cohomology theory for racks and quandles, analogous to group homology. This ultimately led to the current popularity of quandles, since it allowed Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford and Masahico Saito in [**CJK$^+$03**] to define an enhancement of the quandle counting invariant using quandle cocycles, leading to new results about knotted surfaces and more. It was this and subsequent work that led the present authors to study quandles, and ultimately led to this book.

If one restricts oneself to the most important quandle axiom, namely self-distributivity, then one can trace this back to 1880 in the work of Pierce [**Pei80**] where one can read the following comments: *"These are other cases of the distributive principle .... These formulae, which have hitherto escaped notice, are not without interest."* Another early work fully devoted to self-distributivity appeared in 1929 by Burstin and Mayer [**BM29**] dealing with distributive quasigroups: binary algebraic structures in which both right multiplication and left multiplication are bijections, and with the extra property that the operation is left and right distributive on itself (called also Latin quandles).

As quandle theorists, we have found quandle theory not only intrinsically interesting but also very approachable for undergraduates due to its unique mix of geometric pictures and abstract algebra. This book is intended to serve as a text for a one-semester course on quandle theory which might be an upper division math elective or as preparation for a senior thesis in knot theory.

This book assumes that the reader is comfortable with linear algebra and basic set theory but does not assume any previous knowledge of abstract algebra, knot theory or topology. The reader should be familiar with sets, unions, intersections, Cartesian products, functions between sets, injective/surjective/bijective maps as well as vector spaces over fields, linear transformations between vector spaces, and matrix algebra in general. Readers should also be familiar with the integers $\mathbb{Z}$, rationals $\mathbb{Q}$, reals $\mathbb{R}$ and complex numbers $\mathbb{C}$.

The book is organized as follows.

Chapter 1 introduces the basics of knot theory; advanced readers may opt to skip directly to Chapter 2. Chapter 2 introduces important ideas from abstract algebra which are needed for the rest of the book, including introductions to groups, modules, and cohomology assuming only a linear algebra background. Chapter 3 gives a systematic development of the algebraic structures (quandles and kei) arising from oriented and unoriented knots and links, including both theory and practical computations. Chapter 4 looks at important connections between quandles and groups and introduces the basics of algebraic topology, including the fundamental group and the geometric meaning of the fundamental quandle of a knot. In Chapter 5 we look at generalizations of the quandle idea, including racks, bikei, biquandles and biracks. Chapter 6 introduces enhancements of representational knot and link invariants defined from quandles and their generalizations. In Chapter 7 we conclude with applications to generalizations of knots including tangles, knotted surfaces in $\mathbb{R}^4$, and virtual knots.

The authors wish to thank our many students, colleagues and friends without whom this book would not have been possible.

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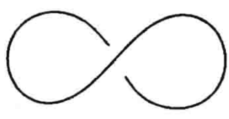
Chapter 1



Knots and Links

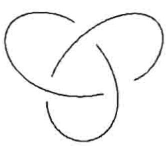
1. Knots and Links

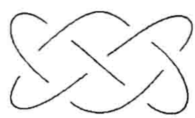
A *knot* is a simple closed curve, where "simple" means the curve does not intersect itself and "closed" means there are no loose ends. We usually think of knots in three-dimensional space since simple closed curves in the line and plane are pretty boring and, perhaps surprisingly, simple closed curves in 4 or more dimensions are also boring, as we will see.

Two knots K_0 and K_1 have the same *knot type* if we can move K_0 around in space in a continuous way, i.e. without cutting or tearing the knot (or the space in which the knot lives!) to match up K_0 with K_1. Formally, K_0 is *ambient isotopic* to K_1 if there is a continuous map $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $H(K_0, 0) = K_0$, $H(K_0, 1) = K_1$ and $H(x,t)$ is injective (one-to-one) for every $t \in [0,1]$. Such a map is called an *ambient isotopy*; if you think of t as a time variable, then H is a movie showing how to continuously deform K_0 onto K_1. If there exists an ambient isotopy H taking K_0 to K_1 we write $H : K_0 \stackrel{\sim}{\to} K_1$.

To specify a knot K we could make a physical model by tying the knot in a rope or cord; a nice trick suggested by Colin Adams in [**Ada04**] is to use an extension cord, so you can join the ends together by plugging the plug into the outlet end.

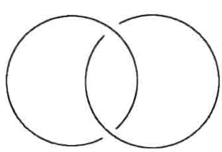
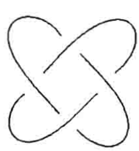
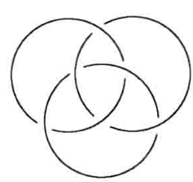
To specify knots in a more print-friendly format, we could give a parametric function $f(t) = (x(t), y(t), z(t))$ where $0 \leq t \leq 1$ and $f(0) = f(1)$. This approach is required in order to study *geometric knot theory*, where the exact positioning of K in space is important. In *topological knot theory*, however, we only care about the position up to ambient isotopy; thus, a simpler solution is to draw pictures or *knot diagrams*. Formally, a *knot diagram* is a projection or shadow of a knot on a plane where we indicate which strand passes over and which passes under at apparent crossing points by drawing the understrand broken.



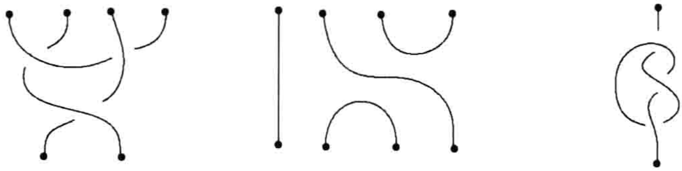


A knot is *tame* if it has a diagram with a finite number of crossing points; knots in which every projection has infinitely many crossing points are called *wild knots*. We will only deal with tame knots in this book.

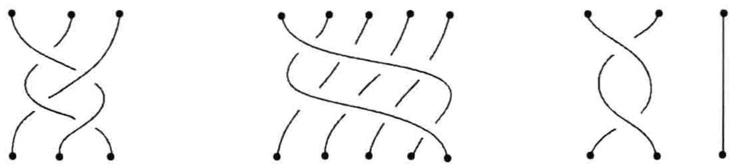
Links, Tangles and Braids (oh my!) There are many kinds of objects related to knots. A *link* consists of several knots possibly linked together; each individual simple closed curve is a *component* of the link. A knot is a link with only one component.

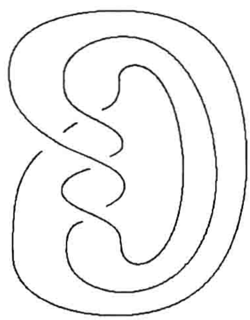
A *tangle* is a portion of a knot or link with fixed endpoints we can think of as inputs and outputs. If there are n inputs and m outputs, we have an (n, m)-tangle.



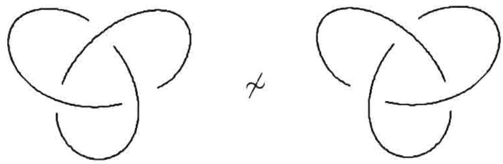
A *braid* is a tangle which has no maxima and no minima in the vertical direction, i.e., a tangle whose strands do not turn around. Note that in any braid, the number of inputs must equal the number of outputs, unlike more general tangles.



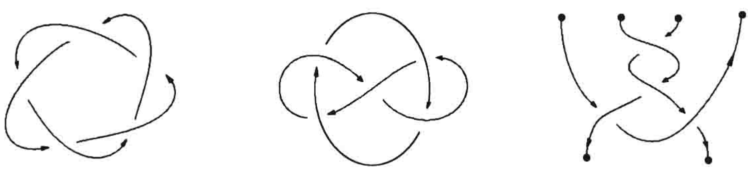
For any braid β there is a knot or link $\hat{\beta}$ called the *closure* of the braid, obtained by joining the top strands to the bottom strands. The converse is also true – every knot or link can be put into braid form, a fact known as *Alexander's Theorem.*

$\beta =$  $\hat{\beta} =$ 

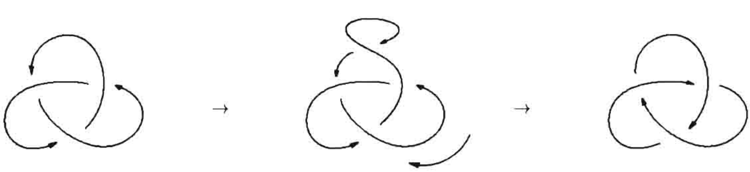
The *obverse* of a knot K is the mirror image of K, denoted $\bar{K}$. A knot may or may not be equivalent to its obverse – the trefoil knot comes in distinct left- and right-handed varieties, for instance. Knots which are different from their obverses are called *chiral*, while knots which are ambient isotopic to their obverses are called *amphichiral.*



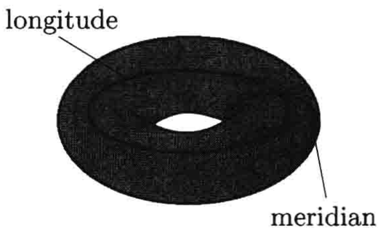
Oriented Knots. For each strand in a knot, link, tangle or braid, we can make a choice of *orientation* or preferred direction of travel. Knots described by a parametrization have an implied orientation in the direction of increasing t value; braids also have an implied orientation of all strands oriented in the same direction (up or down depending on the author's choice of convention). For generic oriented knots, links and tangles, we specify the orientation of each strand with an arrow.



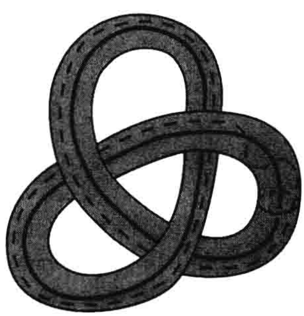
Reversing the orientation of an oriented knot K yields a possibly different oriented knot called the *inverse* or *reverse* of K, denoted $-K$. For two oriented knots K_0 and K_1 to be equivalent, we need an ambient isotopy $H : K_0 \overset{\sim}{\to} K_1$ which respects the orientation of K_1. For example, the trefoil knot K below is equivalent to its inverse $-K$ as illustrated:

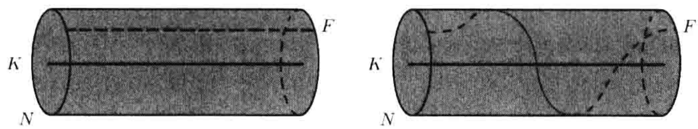


Framed Knots. Like a choice of orientation, a *framing* of a knot is a choice of extra structure we can give to a knot which then must be preserved by an ambient isotopy for two framed knots to be equivalent. Start by inflating the knot K like an inner tube, so we have a knotted solid torus N with K as its core. This solid torus is called a *regular neighborhood* of the knot. A circle on the torus which goes around the torus with the knot is called a *longitude*, while a circle going around a disk slice of the solid torus with the knot at its center is called a *meridian*.



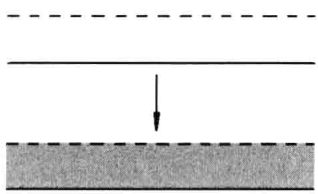
A *framing curve* F is a simple closed curve on the surface of the torus which projects down onto the original knot K in an injective (one-to-one) way, i.e., a longitude of the torus. While F goes around the torus with K exactly once in the longitudinal direction, it can wrap around the meridianal direction of the torus any integer number of times.





Let K be a knot and F a framing curve. A *framed isotopy* of (K_0, F_0) to (K_1, F_1) is an ambient isotopy which carries K_0 to K_1 and carries F_0 to F_1. For a given knot K, with framing curve F, the number of times F wraps meridianally around K (with counterclockwise wraps counted with a positive sign and clockwise twists counted with a minus sign) is called the *framing number* of the framed knot (K, F). For a fixed knot K, two framed knots (K, F_0) and (K, F_1) are framed isotopic only if the framing numbers are equal.

We can think of a framed knot as a 2-component link with the knot and its framing curve forming two sides of a ribbon.



Then, framed isotopy can be understood as movement of the ribbon through space. Similarly a framed link of n components can be understood as an ordinary link of $2n$ components where the components come in parallel pairs forming the two sides of n ribbons, with each of the original n components having its own framing curve and framing number. Similarly, in a framed braid or framed tangle, each strand has its own framing curve and framing number.

We can think of framed isotopy as a mathematical model for knotted 3-dimensional ropes or tori, where ambient isotopy is the model for knotted 1-dimensional curves.