THOMAS JECH

经典名著系列

Set Theory

The Third Millennium Edition, Revised and Expanded

集合论

第3次修订增补版

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The Third Millennium Edition, revised and expanded

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Springer Monographs in Mathematics

For Paula, Pavel, and Susanna

Preface

When I wrote the first edition in the 1970s my goal was to present the state of the art of a century old discipline that had recently undergone a revolutionary transformation. After the book was reprinted in 1997 I started contemplating a revised edition. It has soon become clear to me that in order to describe the present day set theory I would have to write a more or less new book.

As a result this edition differs substantially from the 1978 book. The major difference is that the three major areas (forcing, large cardinals and descriptive set theory) are no longer treated as separate subjects. The progress in past quarter century has blurred the distinction between these areas: forcing has become an indispensable tool of every set theorist, while descriptive set theory has practically evolved into the study of $L(\boldsymbol{R})$ under large cardinal assumptions. Moreover, the theory of inner models has emerged as a major part of the large cardinal theory.

The book has three parts. The first part contains material that every student of set theory should learn and all results contain a detailed proof. In the second part I present the topics and techniques that I believe every set theorist should master; most proofs are included, even if some are sketchy. For the third part I selected various results that in my opinion reflect the state of the art of set theory at the turn of the millennium.

I wish to express my gratitude to the following institutions that made their facilities available to me while I was writing the book: Mathematical Institute of the Czech Academy of Sciences, The Center for Theoretical Study in Prague, CRM in Barcelona, and the Rockefeller Foundation's Bellagio Center. I am also grateful to numerous set theorists who I consulted on various subjects, and particularly to those who made invaluable comments on preliminary versions of the manuscript. My special thanks are to Miroslav Repický who converted the handwritten manuscript to LaTeX. He also compiled the three indexes that the reader will find extremely helpful.

Finally, and above all, I would like to thank my wife for her patience and encouragement during the writing of this book.

Prague, May 2002 *Thomas Jech*

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Part I

Basic Set Theory

1. Axioms of Set Theory

Axioms of Zermelo-Fraenkel

1.1. ***Axiom of Extensionality.*** *If X and Y have the same elements, then $X = Y$.*

1.2. ***Axiom of Pairing.*** *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. ***Axiom Schema of Separation.*** *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. ***Axiom of Union.*** *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. ***Axiom of Power Set.*** *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. ***Axiom of Infinity.*** *There exists an infinite set.*

1.7. ***Axiom Schema of Replacement.*** *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. ***Axiom of Regularity.*** *Every nonempty set has an $\in$-minimal element.*

1.9. ***Axiom of Choice.*** *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

1.10. ***Axiom Schema of Comprehension (false).*** *If P is a property, then there exists a set $Y = \{x : P(x)\}$.*

This principle, however, is false:

1.11. Russell's Paradox. Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the Schema of Comprehension and keep its weak version, the *Schema of Separation*:

If P is a property, then for any X there exists a set $Y = \{x \in X : P(x)\}$.

Once we give up the full Comprehension Schema, Russell's Paradox is no longer a threat; moreover, it provides this useful information: The set of all sets does not exist. (Otherwise, apply the Separation Schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full Comprehension by Separation presents us with a new problem. The Separation Axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

Language of Set Theory, Formulas

The Axiom Schema of Separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation*.

The *formulas* of set theory are built up from the *atomic formulas*

$$x \in y, \qquad x = y$$

by means of *connectives*

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \neg\varphi, \qquad \varphi \rightarrow \psi, \qquad \varphi \leftrightarrow \psi$$

(conjunction, disjunction, negation, implication, equivalence), and *quantifiers*

$$\forall x\, \varphi, \qquad \exists x\, \varphi.$$

In practice, we shall use in formulas other symbols, namely defined predicates, operations, and constants, and even use formulas informally; but it will be tacitly understood that each such formula can be written in a form that only involves $\in$ and $=$ as nonlogical symbols.

Concerning formulas with free variables, we adopt the notational convention that all free variables of a formula

$$\varphi(u_1, \ldots, u_n)$$

are among $u_1, \ldots, u_n$ (possibly some u_i are not free, or even do not occur, in φ). A formula without free variables is called a *sentence*.

Classes

Although we work in ZFC which, unlike alternative axiomatic set theories, has only one type of object, namely sets, we introduce the informal notion of a *class*. We do this for practical reasons: It is easier to manipulate classes than formulas.

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}$$

a *class*. Members of the class C are all those sets x that satisfy $\varphi(x, p_1, \ldots, p_n)$:

$$x \in C \quad \text{if and only if} \quad \varphi(x, p_1, \ldots, p_n).$$

We say that C is *definable from* $p_1, \ldots, p_n$; if $\varphi(x)$ has no parameters p_i then the class C is *definable*.

Two classes are considered equal if they have the same elements: If

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}, \qquad D = \{x : \psi(x, q_1, \ldots, q_m)\},$$

then $C = D$ if and only if for all x

$$\varphi(x, p_1, \ldots, p_n) \leftrightarrow \psi(x, q_1, \ldots, q_m).$$