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Adaptive Computations: Theory and Algorithms

Edited by Tao Tang Jinchao Xu

（自适应计算：理论与算法）

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Preface

Adaptive grid methods are among the most important classes of numerical methods for partial differential equations that arise from scientific and engineering computing. The study of this type of methods has been very active in recent years for algorithm design, theoretical analysis and applications to practical computations. This volume contains a number of self-contained articles for adaptive finite element and finite difference methods, which is aimed to provide some introduction materials for graduate students and junior researchers and a collection of references for researchers and practitioners.

These articles mostly grew out of the lectures notes that were given in *Summer Workshops on Adaptive Method, Theory and Application* organized by Tao Tang, Jinchao Xu and Pingwen Zhang in Peking University, China, during June 20 – August 20, 2005. This summer school was aimed to provide a comprehensive and up-to-date presentation of modern theories and practical applications for adaptive computations. The main lecturers of the Summer School include Weizhang Huang of University of Kansas, Natalia Kopteva of University of Limerick, Zhiping Li of Peking University, John Mackenzie of Strathclyde University, Jinchao Xu of Penn State University, Paul Zegeling of Utrecht University, and Zhimin Zhang of Wayne State University. Other lecturers include Tao Tang, Xiaoping Wang of HKUST, Huazhong Tang and Pingwen Zhang (both from Peking University). More detailed information of this summer school can be found in `http://ccse.pku.edu.cn/activities/2005/adaptiveseminar.htm` (which is most in Chinese). This summer school continued in 2006 but with a more focused program (see `http://ccse.pku.edu.cn/06summerschool/school.html`) and it is expected to continue more in the coming years.

The articles in this volume, which are ordered alphabetically by authors, touch upon various aspects of adaptive methods for algorithmic design, theoretical analysis and practical applications. Chapter 1, by Long Chen and Jinchao Xu, summarizes the basic techniques in local adaptive finite element methods and especially the recent advancements on the convergence and complexity of this type of adaptive methods. Chapter 2, again by Long Chen and Jinchao Xu, describes various postprocessing techniques for gradient recovery that include results on patch symmetric grids, mildly structured grids and general unstructured grids. Chapter 3, by Weichang Huang, covers topics on adaptive algorithms for anisotropic meshes and their theoretical foundations, including basic mathematical principles of mesh adaptation, anisotropic interpolation theory, monitor functions, variational mesh generation, and moving mesh methods. Chapter 4, by Natalia Kopteva, provides a rigorous theoretical analysis on the convergence of moving grid methods for a special class of convection-dominated convection-diffusion problem. Chapter 5, by Zhiping Li, presents a special adaptive grid method, the mesh transformation method, with applications to computation of crystalline and microstructures. Chapter 6, by J.A. Mackenzie and W.R. Mekwi, focuses on moving mesh methods which employ a moving mesh partial differential equation (MMPDE) to evolve the mesh in an appropriate fashion. Chapter 7, by P. A. Zegeling, presents moving grid methods with a summary of many ideas and techniques and applications to reaction-diffusion systems, tumour angiogenesis models, brine transport and magneto-hydrodynamics. Chapter 8, by Zhimin Zhang, discusses two post-processing methods that are instrumental for adaptive finite element methods: the Zienkiewicz-Zhu Superconvergence Patch Recovery (SPR) and recently proposed Polynomial Preserving Recovery (PPR).

It is our great pleasure as editors to thank all of the authors for their hard work to provide us with these high quality articles. We are very grateful for the generous financial supports provided by National Science Foundation of China, The Mathematics Center of Ministry of Education of China, School of Mathematical Sciences of Peking University, and the Joint Research Institute for Applied Mathematics between Peking University and Hong Kong Baptist University. We would like to thank Pingwen Zhang from Peking university for his leadership in helping organize the summer activities in Peking University. We would also like to thank Tammy Lam of Hong Kong Baptist University for the considerable work she put into producing the final layout of the proceedings.

Tao Tang
Hong Kong Baptist University
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October, 2006

Contents

Chapter 1

Convergence of Adaptive Finite Element Methods

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Contents

Adaptive methods are now widely used in the scientific computation to achieve better accuracy with minimum degree of freedom. While these methods have been shown to be very successful, the theory ensuring the convergence of the algorithm and the advantages over non-adaptive methods is still under development. In this chapter, we shall survey recent progress on the convergence analysis of adaptive finite element methods (AFEMs) for second order elliptic partial differential equations [2,11,25,32, 40].

To present the main idea and techniques in their simplest form we restrict ourselves to the linear finite element approximation to the model Poisson equation

$$-\Delta u = f \text{ in } \Omega, \quad \text{and } \ u = 0 \text{ on } \partial\Omega, \tag{1.0.1}$$

posed on a polygonal domain $\Omega \subset \mathbb{R}^2$. We shall prove that there exists an algorithm through the local refinement of triangulations to produce a sequence of approximations of u in an optimal way.

1.1 Introduction

Let $D \subset \mathbb{R}^n$ be a bounded domain. For an integer $k \geqslant 0$, and $1 \leqslant p \leqslant \infty$, $W^{k,p}(D)$ will denote the usual Soblev spaces of functions having distributional derivatives up to order k in L^p. The norm (or semi-norm) is given for $1 \leqslant p < \infty$ by $\|u\|_{k,p,D} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_p^p\right)^{1/p}$ (or $|u|_{k,p,D} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_p^p\right)^{1/p}$) with the usual modification for $p = \infty$. The subscript k, p or D will be omitted if $k = 0$, $p = 2$ or $D = \Omega$. The closure of $C_0^\infty(D)$ with respect to the norm of $W^{k,p}(D)$ is denoted by $W_0^{k,p}(D)$. When $k < 0$, $W^{k,p}(D)$ is defined as the dual space of $W_0^{|k|,p}(D)$. Furthermore $H^k(D) := W^{k,2}(D)$ and $H_0^k(D) := W_0^{k,2}(D)$ are Hilbert spaces in the norm $\|\cdot\|_{k,D}$ or semi-norm $|\cdot|_{k,D}$, respectively.

The letter C or c, with or without subscript, denote generic constants that may not be the same at different occurrences. To avoid writing these constants repeatedly, by $x \lesssim y$ we mean that there exist a constant C such that $x \leqslant Cy$. Obviously $x \gtrsim y$ is defined as $y \lesssim x$, and $x \simeq y$ as $x \lesssim y$ and $x \gtrsim y$. The letter C_i with subscript is used to denote specific important constants.

Finite element methods

The weak formulation of (1.0.1) is: for a given $f \in H^{-1}$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle f, v\rangle \quad \forall v \in H_0^1, \tag{1.1.2}$$

where $a(u, v) = (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v$ and $\langle f, v\rangle$ is the dual pair. In particular when $f \in L^2$, $\langle f, v\rangle = (f, v) = \int_\Omega fv$. The Laplace operator can be understood as the linear operator introduced by the bilinear form $a(\cdot,\cdot)$

$$-\Delta : H_0^1 \mapsto H^{-1}, \quad \text{by } \ \langle -\Delta u, v\rangle = a(u, v), \ \forall v \in H_0^1.$$

By the Poincaré inequality, the bilinear form $a(\cdot,\cdot)$ defines an inner product on V. The existence and uniqueness of the solution to (1.1.2) follows from Riesz representation theorem. Moreover from

$$|u|_1^2 = a(u,u) = \langle f, u\rangle \leqslant \|f\|_{-1}|u|_1,$$

we have the stability result

$$|u|_1 \leqslant \|f\|_{-1}. \tag{1.1.3}$$

Namely the linear operator $\Delta : H_0^1 \mapsto H^{-1}$ is an isomorphism and Δ^{-1} is bounded. Although Δ^{-1} is defined on H^{-1}, we will mainly consider its restriction on $L^2 \subset H^{-1}$. In this case by the regularity theory of elliptic partial differential equations (c.f., [28]), $u = \Delta^{-1} f$ is smoother than H^1 and thus the approximation of u in H^1 norm with certain rate is possible.

The finite element method is to find an approximation by solving (1.1.2) in finite-dimensional subspaces of $H_0^1(\Omega)$ based on triangulations of Ω. That is, given an $f \in L^2$, to find a $u_h \in V^h \subset H_0^1$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V^h. \tag{1.1.4}$$

The existence and uniqueness of the solution to (1.1.4) follows from the Riesz representation theorem since $V^h \subset H_0^1$.

We now discuss the construction of finite element spaces V^h. A triangulation $\mathcal{T}_h$ (also indicated by mesh or grid) of Ω is a *conforming* partition of Ω into a set of triangles. Conforming means the intersection of any two triangles τ_1 and τ_2 in $\mathcal{T}_h$ either consists of a common vertex x_i, edge E or empty. The *interior* node (also indicated by vertices) set and edge set of the triangulation are denoted by $\mathcal{N}_h$ and $\mathcal{E}_h$, respectively. The subscript h here represents the mesh size $h := \max_{\tau\in\mathcal{T}_h} \operatorname{diam}(\tau)$ where $\operatorname{diam}(\tau)$ denotes the diameter of τ. It is always assumed that there are a family of triangulations $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ with $h \to 0$. Properties of triangulations presented below are assumed to hold uniformly with respect to the whole family.

Throughout this article we shall only consider shape regular triangulations. $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is *shape regular* if

$$\max_{\tau\in\mathcal{T}_h} \frac{\operatorname{diam}(\tau)^2}{|\tau|} \leqslant \sigma_s$$

where $|\tau|$ is the area of τ. The shape regularity of triangulations assures that angles of the triangulation remains bounded away from 0 and π which is important to control the interpolation error in H^1 norm [1] and the condition number of the stiffness matrix [27].

For the simplicity, we shall only consider linear finite element space. Given a shape regular triangulation $\mathcal{T}_h$ of Ω, we define

$$V^h := \{v \mid v \in C(\overline{\Omega}), v|_{\partial\Omega} = 0 \text{ and } v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_h\},$$

where $\mathcal{P}_1(\tau)$ denotes the linear polynomial space on τ. It is easily to show that $V^h \subset H_0^1$.

Convergence analysis of finite element methods

Classic convergence analysis of finite element methods are most developed for *quasi-uniform* grids. $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is called *quasi-uniform* if it is shape regular and furthermore satisfies the global assumption

$$\frac{\max_{\tau\in\mathcal{T}_h}|\tau|}{\min_{\tau\in\mathcal{T}_h}|\tau|} \leqslant \sigma_u.$$

Namely all elements of a quasi-uniform grid are around the same size.

For finite element methods there is a natural energy norm defined by the bilinear form $\|u\|_a^2 := a(u,u)$. For this model problem $\|u\|_a = \|\nabla u\| = |u|_1$. The standard convergence analysis of the finite element methods on quasi-uniform meshes has three main ingredients.

1. Quasi-optimality of the finite element approximation:

$$|u - u_h|_1 \lesssim \inf_{v_h\in V^h} |u - v_h|_1.$$

   In particular, $|u - u_h|_1 \lesssim |u - u_I|_1$, where u_I is the nodal interpolation of u based on $\mathcal{T}_h$. Namely $u_I \in V^h$ and $u_I(x_i) = u(x_i)$ for all $x_i \in \mathcal{N}_h$.

2. Interpolation error estimates for u_I on quasi-uniform triangulations $\mathcal{T}_h$:

$$|u - u_I|_1 \lesssim h\|u\|_2, \quad \forall u \in H^2.$$

3. Regularity result of elliptic equations: when the domain Ω is smooth or convex and Lipschitz, $\Delta^{-1} : L^2(\Omega) \mapsto H^2(\Omega) \cap H_0^1(\Omega)$ is a bounded linear operator. That is

$$\|u\|_2 \lesssim \|f\|.$$

Combining 1, 2 and 3, on quasi-uniform triangulations $\mathcal{T}_h$ of a nice domain Ω, we obtain

$$|u - u_h|_1 \lesssim |u - u_I|_1 \lesssim h\|u\|_2 \lesssim h\|f\|.$$

It is easy to see the first order convergence is optimal for general function $u \in H^2$. Indeed for $u = x^2 + y^2$, one can show the convergence rate cannot be greater than one. Thus in the full regularity case the uniform refinement of the triangulations will produce finite element approximations of u in an optimal order.

The regularity result, however, does not hold on general Lipschitz domains. To see this, let us give a simple counter example. Given $\beta \in (0,1)$, let $\Omega = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}$ and let $f = r^\beta \sin(\beta\theta)$ and $u = (1 - r^2)f$. Being the imaginary part of the complex analytic function $z^{1/2}$, f is harmonic in Ω. A direct calculation shows that

$$-\Delta u = 4(1+\beta)f \text{ in } \Omega, \quad \text{and} \quad u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)f \in L^\infty \subset L^2$ but $u \notin H^2$. Furthermore it is easy to see $u \in H^s$, for $0 < s < 1 + \beta$. In views of the approximation theory, we cannot expect approximation rate (in H^1 norm) better than h^β if we insist on quasi-uniform grids.

Equidistribution

To improve the convergence rate, the element size is adapted according to the behavior of the solution. In this case, the element size in areas of the domain where the solution is smooth can stay bounded well away from zero, and thus the global element size is not a good measure of the approximation rate. For this reason, when the optimality of the convergence rate is concerned, $\#\mathcal{T}$, the number of triangles, is used to measure the approximation rate in the setting of adaptive methods that involve local refinement. $\mathcal{T}_N$ is used to denote a triangulation with at most N triangles which is proportional to the number of degree of freedom. Again by default we are considering a sequence of triangulations $\{\mathcal{T}_N\}_{N>N_0}$. Note that $N = O(h^{-2})$ for quasi-uniform grids $\mathcal{T}_h$ when $h \to 0$.

The first order convergence of finite element approximation can be recovered through the correct mesh adaptation. We have the following theorem.

Theorem 1.1. *Let $u \in W^{2,1}(\Omega)$. Suppose a shape regular triangulation $\mathcal{T}_N$ weakly equidistributes the $W^{2,1}$ norm of u in the sense that*

$$|u|_{2,1,\tau} \leqslant \frac{C}{N}|u|_{2,1,\Omega}, \quad \forall \tau \in \mathcal{T}_N. \tag{1.1.5}$$

Then the finite element approximation u_N based on $\mathcal{T}_N$ is of optimal approximation order:

$$|u - u_N|_1 \lesssim N^{-1/2}|u|_{2,1,\Omega}. \tag{1.1.6}$$

Proof. Using the embedding: $W^{2,1}(\Omega) \subset H^1(\Omega) \cap C(\overline{\Omega})$, we know the nodal interpolation u_I is well defined and

$$|u - u_I|_{1,\tau} \leqslant C\|u\|_{2,1,\tau}, \quad \forall \tau \in \mathcal{T}_N.$$

Since the nodal interpolation preserve linear polynomials, by Bramble-Hilbert lemma, we have

$$|u - u_I|_{1,\tau} \leqslant C|u|_{2,1,\tau}, \quad \forall \tau \in \mathcal{T}_N.$$

We emphasis that constants in the above inequalities do not depends on the element τ since those norms are scaling invariant. Squaring and summing over all the elements, we get

$$|u - u_I|_1^2 \lesssim \sum_{\tau \in \mathcal{T}_N} |u|_{2,1,\tau}^2 \lesssim N^{-1}|u|_{2,1,\Omega}^2.$$

In the last step we have used the equidistribution assumption (1.1.5). □

We would like to stress two important points contained in Theorem 1.1. First we use a weaker norm $W^{2,1}$ of u to obtain the optimal convergence order $N^{-1/2} = \mathcal{O}(h)$. We still use the second derivative of u but the norm is shifted from L^2 norm to a weaker one L^1. In general we may need to work on Besov spaces. We refer the reader to [10, 24, 41] for the definition and properties of Besov space. Here we only remark that such spaces are function spaces suitable for the theory of nonlinear approximation by piecewise polynomials. The regularity theory of elliptic partial differential equation in terms of Besov norms can be found at [3, 4, 20, 21]. It is shown in those works that the solution to the Poisson's equation indeed has typically higher Besov than Sobolev regularity and thus the use of adaptive scheme gives a better asymptotic approximation rate.

Secondly an idea case of (1.1.5) is the equidistribution principal widely used in the literature:

$$|u|_{2,1,\tau} = \frac{1}{N}|u|_{2,1,\Omega}, \quad \forall \tau \in \mathcal{T}_N.$$

Namely a good mesh adaptation is to equidistribute some local error bound. We would like to elaborate that, in the current setting, equidistribution is indeed a sufficient condition for optimal error, but by no means this has to be a necessary condition. Namely the equidistribution principal can be swerely violated but asymptoticly optimal error estimates can still be maintained. For example if a bounded number of