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C^* -Algebras
Volume 3: General Theory
of C^* -Algebras

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C*-Algebras

Volume 3: General Theory of C*-Algebras

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C*-ALGEBRAS

**VOLUME 3:
GENERAL THEORY OF C*-ALGEBRAS**

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Preface

Functional analysis plays an important role in the program of studies at the Swiss Federal Institute of Technology. At present, courses entitled Functional Analysis I and II are taken during the fifth and sixth semester, respectively. I have taught these courses several times and after a while typewritten lecture notes resulted that were distributed to the students. During the academic year 1987/88, I was fortunate enough to have an eager enthusiastic group of students that I had already encountered previously in other lecture courses. These students wanted to learn more in the area and asked me to design a continuation of the courses. Accordingly, I proceeded during the academic year, following, with a series of special lectures, Functional Analysis III and IV, for which I again distributed typewritten lecture notes. At the end I found that there had accumulated a mass of textual material, and I asked myself if I should not publish it in the form of a book. Unfortunately, I realized that the two special lecture series (they had been given only once) had been badly organized and contained material that should have been included in the first two portions. And so I came to the conclusion that I should write everything anew – and if at all – then preferably in English. Little did I realize what I was letting myself in for! The number of pages grew almost imperceptibly and at the end it had more than doubled. Also, the English language turned out to be a stumbling block for me; I would like to take this opportunity to thank Prof. Imre Bokor and Prof. Edgar Reich for their help in this regard. Above all I must thank Mrs. Barbara Aquilino, who wrote, first a WordMARCTM, and then a L^AT_EXTM version with great competence, angelic patience, and utter devotion, in spite of illness. My thanks also go to the Swiss Federal Institute of Technology that generously provided the infrastructure for this extensive enterprise and to my colleagues who showed their understanding for it.

Corneliu Constantinescu

Introduction

This book has evolved from the lecture course on Functional Analysis I had given several times at the ETH. The text has a strict logical order, in the style of “Definiton – Theorem – Proof – Example – Exercises”. The proofs are rather thorough and there are many examples.

The first part of the book (the first three chapters, resp. the first two volumes) is devoted to the theory of Banach spaces in the most general sense of the term. The purpose of the first chapter (resp. first volume) is to introduce those results on Banach spaces which are used later or which are closely connected with the book. It therefore only contains a small part of the theory, and several results are stated (and proved) in a diluted form. The second chapter (which together with Chapter 3 makes the second volume) deals with Banach algebras (and involutive Banach algebras), which constitute the main topic of the first part of the book. The third chapter deals with compact operators on Banach spaces and linear (ordinary and partial) differential equations – applications of the theory of Banach algebras.

The second part of the book (the last four chapters, resp. the last three volumes) is devoted to the theory of Hilbert spaces, once again in the general sense of the term. It begins with a chapter (Chapter 4, resp. Volume 3) on the theory of C^* -algebras and W^* -algebras which are essentially the focus of the book. Chapter 5 (resp. Volume 4) treats Hilbert spaces for which we had no need earlier. It contains the representation theorems, i.e. the theorems on isometries between abstract C^* -algebras and the concrete C^* -algebras of operators on Hilbert spaces. Chapter 6 (which together with Chapter 7 makes Volume 5) presents the theory of \mathcal{L}^p -spaces of operators, its application to the self-adjoint linear (ordinary and partial) differential equations, and the von Neumann algebras. Finally, Chapter 7 presents examples of C^* -algebras defined with the aid of groups, in particular the Clifford algebras. Many important domains of C^* -algebras are ignored in the present book. It should be emphasized that the whole theory is constructed in parallel for the real and for the complex numbers, i.e. the C^* -algebras are real or complex.

In addition to the above (vertical) structure of the book, there is also a second (horizontal) division. It consists of a main strand, eight branches, and additional material. The results belonging to the main strand are marked with (0). Logically speaking, a reader could restrict himself/herself to these and ignore the rest. Results on the eight subsidiary branches are marked with (1), (2), (3), (4), (5), (6), (7), and (8). The key is

1. Infinite Matrices
2. Banach Categories
3. Nuclear Maps
4. Locally Compact Groups
5. Differential Equations
6. Laurent Series
7. Clifford Algebras
8. Hilbert C^* -Modules

These are (logically) independent of each other, but all depend on the main strand. Finally, the results which belong to the additional material have no marking and – from a logical perspective – may be ignored. So the reader can shorten for himself/herself this very long book using the above marks. Also, since the proofs are given with almost all references, it is possible to get into the book at any level and not to read it linearly.

We assume that the reader is familiar with classical analysis and has rudimentary knowledge of set theory, linear algebra, point-set topology, and integration theory. The book addresses itself mainly to mathematicians, or to physicists interested in C^* -algebras.

I would like to apologize for any omissions in citations occasioned by the fact that my acquaintance with the history of functional analysis is, unfortunately, very restricted. For this history we recommend the following texts.

1. BIRKHOFF, G. and KREYSZIG, E., The Establishment of Functional Analysis, *Historia Mathematica* 11 (1984), 258–321.
2. BOURBAKI, N., Elements of the History of Mathematics, (21. Topological Vector Spaces), Springer-Verlag (1994).
3. DIEUDONNÉ, J., History of Functional Analysis, North-Holland (1981).
4. DIEUDONNÉ, J., A Panorama of Pure Mathematics (Chapter C III: Spectral Theory of Operators), Academic Press (1982).
5. HEUSER, H., Funktionalanalysis, 2. Auflage (Kapitel XIX: Ein Blick auf die werdende Funktionalanalysis), Teubner (1986), 3. Auflage (1992).
6. KADISON, R.V., Operator Algebras, the First Forty Years, in: Proceedings of Symposia in Pure Mathematics 38 I (1982), 1–18.
7. MONNA, A.F., Functional Analysis in Historical Perspective, John Wiley & Sons (1973).

8. STEEN, L.A., Highlights in the History of Spectral Theory, Amer. Math. Monthly 80 (1973), 359–382.

There is no shortage of excellent books on C^* -algebras. Nevertheless, we hope that this book will be also of some utility to the mathematics community.

Table of Contents of Volume 3

Introduction	xix
4 C^* -Algebras	3
4.1 The General Theory	3
4.1.1 General Results	4
4.1.2 The Symmetry of C^* -Algebra	30
4.1.3 Functional calculus in C^* -Algebras	56
4.1.4 The Theorem of Fuglede–Putnam	75
4.2 The Order Relation	92
4.2.1 Definition and General Properties	92
4.2.2 More about the Order Relation	101
4.2.3 Examples	116
4.2.4 Powers of Positive Elements	123
4.2.5 The Modulus	143
4.2.6 Ideals and Quotients of C^* -Algebras	150
4.2.7 The Ordered Set of Orthogonal Projections	162
4.2.8 Approximate Unit	178
4.3 Supplementary Results on C^* -Algebras	208
4.3.1 The Exterior Multiplication	208
4.3.2 Order Complete C^* -Algebras	215
4.3.3 The Carrier	243
4.3.4 Hereditary C^* -Subalgebras	263
4.3.5 Simple C^* -algebras	276
4.3.6 Supplementary Results Concerning Complexification	286
4.4 W^* -Algebras	297
4.4.1 General Properties	297
4.4.2 F as an E -submodule of E'	309
4.4.3 Polar Representation	335
4.4.4 W^* -Homomorphisms	361
Name Index	385

Subject Index	388
Symbol Index	411

Contents of All Volumes

Table of Contents of Volume 1

Introduction	xix
Some Notation and Terminology	1
1 Banach Spaces	7
1.1 Normed Spaces	7
1.1.1 General Results	7
1.1.2 Some Standard Examples	12
1.1.3 Minkowski's Theorem	31
1.1.4 Locally Compact Normed Spaces	35
1.1.5 Products of Normed Spaces	37
1.1.6 Summable Families	40
Exercises	58
1.2 Operators	61
1.2.1 General Results	61
1.2.2 Standard Examples	74
1.2.3 Infinite Matrices	92
1.2.4 Quotient Spaces	113
1.2.5 Complemented Subspaces	123
1.2.6 The Topology of Pointwise Convergence	134
1.2.7 Convex Sets	138
1.2.8 The Alaoglu–Bourbaki Theorem	148
1.2.9 Bilinear Maps	150
Exercises	153
1.3 The Hahn–Banach Theorem	159
1.3.1 The Banach Theorem	159
1.3.2 Examples in Measure Theory	171
1.3.3 The Hahn–Banach Theorem	180
1.3.4 The Transpose of an Operator	191

1.3.5	Polar Sets	199
1.3.6	The Bidual	211
1.3.7	The Krein–Šmulian Theorem	228
1.3.8	Reflexive Spaces	240
1.3.9	Completion of Normed Spaces	245
1.3.10	Analytic Functions	246
	Exercises	254
1.4	Applications of Baire’s Theorem	256
1.4.1	The Banach–Steinhaus Theorem	256
1.4.2	Open Mapping Principle	264
	Exercises	280
1.5	Banach Categories	281
1.5.1	Definitions	281
1.5.2	Functors	288
1.6	Nuclear Maps	308
1.6.1	General Results	308
1.6.2	Examples	322
1.7	Ordered Banach spaces	334
1.7.1	Ordered normed spaces	334
1.7.2	Order Continuity	340
	Name Index	357
	Subject Index	359
	Symbol Index	371

Table of Contents of Volume 2

Introduction	xix
2 Banach Algebras	3
2.1 Algebras	3
2.1.1 General Results	3
2.1.2 Invertible Elements	13
2.1.3 The Spectrum	17
2.1.4 Standard Examples	32
2.1.5 Complexification of Algebras	51
Exercises	65
2.2 Normed Algebras	69
2.2.1 General Results	69
2.2.2 The Standard Examples	82
2.2.3 The Exponential Function and the Neumann Series	114
2.2.4 Invertible Elements of Unital Banach Algebras	125
2.2.5 The Theorems of Riesz and Gelfand	153
2.2.6 Poles of Resolvents	161
2.2.7 Modules	174
Exercises	197
2.3 Involutive Banach Algebras	201
2.3.1 Involutive Algebras	201
2.3.2 Involutive Banach Algebras	241
2.3.3 Sesquilinear Forms	275
2.3.4 Positive Linear Forms	287
2.3.5 The State Space	305
2.3.6 Involutive Modules	322
Exercises	328
2.4 Gelfand Algebras	331
2.4.1 The Gelfand Transform	331
2.4.2 Involutive Gelfand Algebras	343

2.4.3	Examples	358
2.4.4	Locally Compact Additive Groups	365
2.4.5	Examples	378
2.4.6	The Fourier Transform	390
	Exercises	396
3	Compact Operators	399
3.1	The General Theory	399
3.1.1	General Results	399
3.1.2	Examples	419
3.1.3	Fredholm Operators	437
3.1.4	Point Spectrum	468
3.1.5	Spectrum of a Compact Operator	477
3.1.6	Integral Operators	489
	Exercises	517
3.2	Linear Differential Equations	518
3.2.1	Boundary Value Problems for Differential Equations . . .	518
3.2.2	Supplementary Results	530
3.2.3	Linear Partial Differential Equations	549
	Exercises	563
	Name Index	565
	Subject Index	568
	Symbol Index	588

Table of Contents of Volume 3

Introduction	xix
4 C^* -Algebras	3
4.1 The General Theory	3
4.1.1 General Results	4
4.1.2 The Symmetry of C^* -Algebra	30
4.1.3 Functional calculus in C^* -Algebras	56
4.1.4 The Theorem of Fuglede–Putnam	75
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4.2.3 Examples	116
4.2.4 Powers of Positive Elements	123
4.2.5 The Modulus	143
4.2.6 Ideals and Quotients of C^* -Algebras	150
4.2.7 The Ordered Set of Orthogonal Projections	162
4.2.8 Approximate Unit	178
4.3 Supplementary Results on C^* -Algebras	208
4.3.1 The Exterior Multiplication	208
4.3.2 Order Complete C^* -Algebras	215
4.3.3 The Carrier	243
4.3.4 Hereditary C^* -Subalgebras	263
4.3.5 Simple C^* -algebras	276
4.3.6 Supplementary Results Concerning Complexification	286
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4.4.4 W^* -Homomorphisms	361
Name Index	385

Subject Index	388
Symbol Index	411