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William Arveson

An Invitation to C*-Algebras

C*代数入门

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An Invitation to C^*-Algebras

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Preface

This book gives an introduction to C^*-algebras and their representations on Hilbert spaces. We have tried to present only what we believe are the most basic ideas, as simply and concretely as we could. So whenever it is convenient (and it usually is), Hilbert spaces become separable and C^*-algebras become GCR. This practice probably creates an impression that nothing of value is known about other C^*-algebras. Of course that is not true. But insofar as representations are concerned, we can point to the empirical fact that to this day no one has given a concrete parametric description of even the irreducible representations of any C^*-algebra which is not GCR. Indeed, there is metamathematical evidence which strongly suggests that no one ever will (see the discussion at the end of Section 3.4). Occasionally, when the idea behind the proof of a general theorem is exposed very clearly in a special case, we prove only the special case and relegate generalizations to the exercises.

In effect, we have systematically eschewed the Bourbaki tradition.

We have also tried to take into account the interests of a variety of readers. For example, the multiplicity theory for normal operators is contained in Sections 2.1 and 2.2. (it would be desirable but not necessary to include Section 1.1 as well), whereas someone interested in Borel structures could read Chapter 3 separately. Chapter 1 could be used as a bare-bones introduction to C^*-algebras. Sections 2.1 and 2.3 together contain the basic structure theory for type I von Neumann algebras, and are also largely independent of the rest of the book.

The level of exposition should be appropriate for a second year graduate student who is familiar with the basic results of functional analysis, measure theory, and Hilbert space. For example, we assume the reader knows the Hahn – Banach theorem, Alaoglu's theorem, the Krein – Milman theorem, the spectral theorem for normal operators, and the elementary theory of commutative Banach algebras. On the other hand, we have avoided making use of dimension theory and most of the more elaborate machinery of reduction theory (though we do use the notation for direct integrals in Sections 2.2 and 4.3). More regrettably, some topics have been left out merely to keep down the size of the book; for example, applications to the theory of unitary representations of locally compact groups are barely mentioned. To fill in these many gaps, the reader should consult the comprehensive monographs of Dixmier [6, 7].

A preliminary version of this manuscript was finished in 1971, and during the subsequent years was widely circulated in preprint form under the title *Representations of C^*-algebras*. The present book has been reorganized, and new material has been added to correct what we felt were serious omissions in the earlier version. It has been used as the basis for lectures in Berkeley and in Aarhus.

We are indebted to many colleagues and students who read the manuscript, pointed out errors, and offered constructive criticism. Special thanks go to Cecelia Bleecker, Larry Brown, Paul Chernoff, Ron Douglas, Dick Loebl, Donal O'Donovan, Joan Plastiras, and Erling Størmer.

This subject has more than its share of colorless and obscure terminology. In particular, one always has to choose between calling a C^*-algebra GCR, type I, or postliminal. The situation is no better in French: does postliminaire mean post-preliminary? In this book we have reverted to Kaplansky's original acronym, simply because it takes less space to write. More sensibly, we have made use of Halmos' symbol □ to signal the end of a proof.

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Contents

Fundamentals 1

This chapter contains what we consider to be the essentials of noncommutative C^*-algebra theory. This is the material that anyone who wants to work seriously with C^*-algebras needs to know. The most tractable C^*-algebras are those that can be related to compact operators in a certain specific way. These are the so-called GCR algebras, and they are introduced in Section 1.5, after a rather extensive discussion of C^*-algebras of compact operators in Section 1.4.

Representations are first encountered in Section 1.3; they remain near the center of discussion throughout the chapter, and indeed throughout the remainder of the book (excepting Chapter 3).

1.1. Operators and C^*-algebras

A *C^*-algebra of operators* is a subset $\mathscr{A}$ of the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on a Hilbert space $\mathscr{H}$, which is closed under all of the algebraic operations on $\mathscr{L}(\mathscr{H})$ (addition, multiplication, multiplication by complex scalars), is closed in the norm topology of $\mathscr{L}(\mathscr{H})$, and most importantly is closed under the adjoint operation $T \mapsto T^*$ in $\mathscr{L}(\mathscr{H})$. Every operator T on $\mathscr{H}$ determines a C^*-algebra $C^*(T)$, namely the smallest C^*-algebra containing both T and the identity. It is more or less evident that $C^*(T)$ is the norm closure of all polynomials $p(T, T^*)$, where $p(x, y)$ ranges over all polynomials in the two free (i.e., noncommuting) variables x and y having complex coefficients. However since T and T^* do not generally commute, these polynomials in T and T^* are of little use in answering questions, and in particular the above remark sheds no light on the structure of $C^*(T)$. Nevertheless, $C^*(T)$ contains much information about T, and one could view this book as a description of what that information is and how one goes about extracting it.

We will say that two operators S and T (acting perhaps, on different Hilbert spaces) are *algebraically equivalent* if there is a $*$-isomorphism (that is, an isometric $*$-preserving isomorphism) of $C^*(S)$ onto $C^*(T)$ which carries S into T. Note that this is more stringent than simply requiring that $C^*(S)$ and $C^*(T)$ be $*$-isomorphic. We will see presently that two normal operators are algebraically equivalent if and only if they have the same spectrum; thus one may think of algebraically equivalent nonnormal operators as having the same "spectrum" in some generalized sense, which will be made more precise in Chapter 4.

We now collect a few generalities. A (general) *C^*-algebra* is a Banach algebra A having an involution $*$ (that is, a conjugate-linear map of A into itself satisfying $x^{**} = x$ and $(xy)^* = y^*x^*$, $x, y \in A$) which satisfies $\|x^*x\| = \|x\|^2$ for all $x \in A$. It is very easy to see that a C^*-algebra of operators on a Hilbert space is a C^*-algebra, and we will eventually prove a theorem of Gelfand and Naimark which asserts the converse: every C^*-algebra is isometrically $*$-isomorphic with a C^*-algebra of operators on a Hilbert space (Theorem 1.7.3).

Let A be a *commutative* C^*-algebra. Then in particular A is a commutative Banach algebra, and therefore the set of all nonzero complex homomorphisms of A is a locally compact Hausdorff space in its usual topology. This space will be called the *spectrum* of A, and it is written $\hat{A}$. A standard result asserts that $\hat{A}$ is compact iff A contains a multiplicative identity. Now the Gelfand map is generally a homomorphism of A into the Banach algebra $C(\hat{A})$ of all continuous complex valued functions on $\hat{A}$ vanishing at ∞. In this case, however, much more is true.

Theorem 1.1.1. *The Gelfand map is an isometric $*$-isomorphism of A onto $C(\hat{A})$.*

Here, the term $*$-isomorphism means that, in addition to the usual properties of an isomorphism, $x^* \in A$ gets mapped into the complex conjugate of the image of x. We will give the proof of this theorem for the case where A contains an identity 1; the general case follows readily from this by the process of adjoining an identity (Exercise 1.1.H).

First, let $\omega \in \hat{A}$. Then we claim $\omega(x^*) = \overline{\omega(x)}$ for all $x \in A$. This reduces to proving that $\omega(x)$ is real for all $x = x^*$ in A (since every $x \in A$ can be written $x = x_1 + ix_2$, with $x_i = x_i^* \in A$). Therefore choose $x = x^* \in A$, and for every real number t define $u_t = e^{itx}$ (for any element $z \in A$, e^z is defined by the convergent power series $\sum_{n=0}^{\infty} z^n/n!$, and the usual manipulations show that $e^{z+w} = e^z e^w$ since z and w commute). By examining the power series we see that $u_t^* = e^{-itx}$, and hence $u_t^* u_t = e^{-itx+itx} = 1$. Thus $\|u_t\|^2 = \|u_t^* u_t\| = \|1\| = 1$, and since the complex homomorphism ω has norm 1 we conclude $\exp t\, \mathcal{R}e\; i\omega(x) = |e^{\omega(itx)}| = |\omega(u_t)| \leq 1$, for all $t \in \mathbb{R}$. This can only mean $\mathcal{R}e\; i\omega(x) = 0$, and hence $\omega(x)$ is real.

Now let $\gamma(x)$ denote the image of x in $C(\hat{A})$, i.e., $\gamma(x)(\omega) = \omega(x)$, $\omega \in \hat{A}$. Then we have just proved $\gamma(x^*) = \overline{\gamma(x)}$, and we now claim $\|\gamma(x)\| = \|x\|$. Indeed the left side is the spectral radius of x which, by the Gelfand-Mazur theorem, is $\lim_n \|x^n\|^{1/n}$. But if $x = x^*$ then we have $\|x\|^2 = \|x^*x\| = \|x^2\|$; replacing x with x^2 gives $\|x\|^4 = \|x^2\|^2 = \|x^4\|$, and so on inductively, giving $\|x\|^{2^n} = \|x^{2^n}\|$, $n \geqslant 1$. This proves $\|x\| = \lim_n \|x^n\|^{1/n}$ if $x = x^*$, and the case of general x reduces to this by the trick $\|\gamma(x)\|^2 = \|\overline{\gamma(x)}\gamma(x)\| = \|\gamma(x^*x)\| = \|x^*x\| = \|x\|^2$, applying the above to the self-adjoint element x^*x.

Thus γ is an isometric $*$-isomorphism of A onto a closed self-adjoint subalgebra of $C(\hat{A})$ containing 1; since $\gamma(A)$ always separates points, the proof is completed by an application of the Stone–Weierstrass theorem. □

An element x of a C^*-algebra is called *normal* if $x^*x = xx^*$. Note that this is equivalent to saying that the sub C^*-algebra generated by x is commutative.

Corollary. *If x is a normal element of a C^*-algebra with identity, then the norm of x equals its spectral radius.*

Proof. Consider x to be an element of the commutative C^*-algebra it generates (together with the identity). Then the assertion follows from the fact that the Gelfand map is an isometry. □

Theorem 1.1.1 is sometimes called the *abstract spectral theorem*, since it provides the basis for a powerful functional calculus in C^*-algebras. In order to discuss this, let us first recall that if B is a Banach subalgebra of a Banach algebra A with identity 1, such that $1 \in B$, then an element x in B has a spectrum $\mathrm{sp}_A(x)$ relative to A as well as a spectrum $\mathrm{sp}_B(x)$ relative to B, and in general one has $\mathrm{sp}_A(x) \subseteq \mathrm{sp}_B(x)$. Of course, the inclusion is often proper. But if A is a C^*-algebra and B is a C^*-subalgebra, then the two spectra must be the same. To indicate why this is so, we will show that if $x \in B$ is invertible in A, then x^{-1} belongs to B (a moment's thought shows that the assertion reduces to this). For that, note that x^* is invertible, and since the element $(x^*x)^{-1}x^*$ is clearly a *left* inverse for x, we must have $x^{-1} = (x^*x)^{-1}x^*$. So to prove that $x^{-1} \in B$, it suffices to show that $(x^*x)^{-1} \in B$. Actually, we will show that x^*x is invertible in the still smaller C^*-algebra B_0 generated by x^*x and e. For since B_0 is commutative, 1.1.1 implies that the spectrum (relative to B_0) of the self-adjoint element x^*x is real, and in particular this relative spectrum is its own boundary, considered as a subset of the complex plane. By the spectral permanence theorem ([23], p. 33), the latter coincides with $\mathrm{sp}_A(x^*x)$. Because $0 \notin \mathrm{sp}_A(x^*x)$, we conclude that x^*x is invertible in B_0.

These remarks show in particular that it is unambiguous to speak of the spectrum of an operator T on a Hilbert space $\mathscr{H}$, so long as it is taken relative to a C^*-algebra. Thus, the spectrum of T in the traditional sense (i.e., relative to $\mathscr{L}(\mathscr{H})$) is the same as the spectrum of T relative to the subalgebra $C^*(T)$. They also show that the spectrum of a self-adjoint element of an arbitrary C^*-algebra (commutative or not) is always real.

We can now deduce the functional calculus for normal elements of C^*-algebras. Fix such an element x in a C^*-algebra with identity, and let B be the C^*-algebra generated by x and e. Define a map of $\hat{B}$ into $\mathbb{C}$ as follows: $\omega \to \omega(x)$. This is continuous and 1—1, thus since $\hat{B}$ is compact it is a homeomorphism of $\hat{B}$ onto its range. By the preceding discussion the range of this map is $\operatorname{sp}_A(x) = \operatorname{sp}(x)$. So this map induces, by composition, an isometric $*$-isomorphism of $C(\operatorname{sp}(x))$ onto B. It is customary to write the image of a function $f \in C(\operatorname{sp}(x))$ under this isomorphism as $f(x)$. Note that the formula suggested by this notation reduces to the expected thing when f is a polynomial in ζ and $\bar{\zeta}$; for example, if $f(\zeta) = \zeta^2\bar{\zeta}$ then $f(x) = x^2x^*$. This process of "applying" continuous functions on $\operatorname{sp}(x)$ to x is called the *functional calculus*.

In particular, when T is a normal operator on a Hilbert space we have defined expressions of the form $f(T)$, $f \in C(\operatorname{sp}(T))$. In this concrete setting one can even extend the functional calculus to arbitrary bounded (or even unbounded) Borel functions defined on $\operatorname{sp}(T)$, but we shall have no particular need for that in this book. It is now a simple matter to prove:

Theorem 1.1.2. *Let S and T be normal operators. Then S and T are algebraically equivalent if, and only if, they have the same spectrum.*

PROOF. Assume first that $\operatorname{sp}(S) = \operatorname{sp}(T)$. Then by the above we have $\|f(S)\| = \sup\{|f(z)| : z \in \operatorname{sp}(S)\} = \|f(T)\|$, for every continuous function f on $\operatorname{sp}(S)$. This shows that the map $\phi: f(S) \to f(T)$, $f \in C(\operatorname{sp}(S))$, is an isometric $*$-isomorphism of $C^*(S)$ on $C^*(T)$ which carries S to T. Conversely, if such a ϕ exists, then the spectrum of S relative to $C^*(S)$ must equal the spectrum of $\phi(S) = T$ relative to $C^*(T)$. By the preceding remarks, this implies $\operatorname{sp}(S) = \operatorname{sp}(T)$. □

EXERCISES

1.1.A. Let e be an element of a C^*-algebra which satisfies $ex = x$ for every $x \in A$. Show that e is a unit, $e = e^*$, and $\|e\| = 1$.

1.1.B. Let A be a Banach algebra having an involution $x \to x^*$ which satisfies $\|x\|^2 \leqslant \|x^*x\|$ for every x. Show that A is a C^*-algebra.

1.1.C. (Mapping theorem.) Let x be a self-adjoint element of a C^*-algebra with unit and let $f \in C(\operatorname{sp}(x))$. Show that the spectrum of $f(x)$ is $f(\operatorname{sp}(x))$.

1.1.D. Let A be the algebra of all continuous complex-valued functions, defined on the closed disc $D = \{|z| \leqslant 1\}$ in the complex plane, which are analytic in the interior of D.

a. Show that A is a commutative Banach algebra with unit, relative to the norm $\|f\| = \sup_{|z| \leqslant 1} |f(z)|$.

b. Show that $f^*(z) = \overline{f(\bar{z})}$ defines an isometric involution in A.

c. Show that not every complex homomorphism ω of A satisfies $\omega(f^*) = \overline{\omega(f)}$.

1.1.E. Let A be a C^*-algebra without unit. Show that, for every x in A:

$$\|x\| = \sup_{\|y\| \leq 1} \|xy\|.$$

1.1.F. Let S and T be normal operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. Show that $C^*(S)$ is $*$-isomorphic to $C^*(T)$ iff $\mathrm{sp}(S)$ is homeomorphic to $\mathrm{sp}(T)$.

1.1.G. Let $f:\mathbb{R} \to \mathbb{C}$ be a continuous function and let A be a C^*-algebra with unit. Show that the mapping $x \mapsto f(x)$ is a continuous function from $\{x \in A : x = x^*\}$ into A.

1.1.H. (Exercise on adjoining a unit.) Let A be a C^*-algebra without unit, and for each x in A let L_x be the linear operator on A defined by $y \mapsto xy$. Let B be the set of all operators on A of the form $\lambda 1 + L_x$, $\lambda \in \mathbb{C}$, $x \in A$.

Show that B is a C^*-algebra with unit relative to the operator norm and the involution $(\lambda 1 + L_x)^* = \bar{\lambda} 1 + L_{x^*}$, and that $x \mapsto L_x$ is an isometric $*$-isomorphism of A onto a closed ideal in B of codimension 1. [*Hint*: use 1.1.B.]

1.1.I. Discuss briefly how the functional calculus (for self-adjoint elements) must be modified for C^*-algebras with no unit. In particular, explain why $\sin x$ makes sense for every self-adjoint element x but $\cos x$ does not. [*Hint*: use 1.1.H to define the spectrum of an element in a non-unital C^*-algebra.]

1.2. Two Density Theorems

There are two technical results which are extremely useful in dealing with $*$-algebras of operators. We will discuss these theorems in this section and draw out a few applications.

The *null space* of a set $\mathscr{S} \subseteq \mathscr{L}(\mathscr{H})$ of operators is the closed subspace of all vectors $\xi \in \mathscr{H}$ such that $S\xi = 0$ for all $S \in \mathscr{S}$. The *commutant* of $\mathscr{S}$ (written $\mathscr{S}'$) is the family of operators which commute with each element of $\mathscr{S}$. Note that $\mathscr{S}'$ is always closed under the algebraic operations, contains the identity operator, and is closed in the weak operator topology. Moreover, if $\mathscr{S}$ is *self-adjoint*, that is $\mathscr{S} = \mathscr{S}^*$ is closed under the $*$-operation, then so is $\mathscr{S}'$. Now it is evident that $\mathscr{S}$ is always contained in $\mathscr{S}''$, but even when $\mathscr{S}$ is a weakly closed algebra containing the identity the inclusion may be proper. According to the following celebrated theorem of von Neumann, however, one has $\mathscr{S} = \mathscr{S}''$ if in addition $\mathscr{S}$ is self-adjoint.

Theorem 1.2.1 Double commutant theorem. *Let $\mathscr{A}$ be a self-adjoint algebra of operators which has trivial null space. Then $\mathscr{A}$ is dense in $\mathscr{A}''$ in both the strong and the weak operator topologies.*

Proof. Let $\mathscr{A}_w$ and $\mathscr{A}_s$ denote the weak and strong closures of $\mathscr{A}$. Then clearly $\mathscr{A}_s \subseteq \mathscr{A}_w \subseteq \mathscr{A}''$, and it suffices to show that each operator $T \in \mathscr{A}''$ can be strongly approximated by operators in $\mathscr{A}$; that is, for every $\varepsilon > 0$, every $n = 1, 2, \ldots$, and every choice of n vectors $\xi_1, \xi_2, \ldots, \xi_n \in \mathscr{H}$, there is an operator $S \in \mathscr{A}$ such that $\sum_{k=1}^{n} \|T\xi_k - S\xi_k\|^2 < \varepsilon^2$.

Consider first the case $n = 1$, and let P be the projection onto the closed subspace $[\mathscr{A}\xi_1]$. Note first that P commutes with $\mathscr{A}$. Indeed the range of P is invariant under $\mathscr{A}$; since $\mathscr{A} = \mathscr{A}^*$, so is the range of $P^\perp = I - P$, and this implies $P \in \mathscr{A}'$. Next observe that $\xi_1 \in [\mathscr{A}\xi_1]$, or equivalently, $P^\perp\xi_1 = 0$. For if $S \in \mathscr{A}$ then $SP^\perp\xi_1 = P^\perp S\xi_1 = 0$ (because $S\xi_1 \in [\mathscr{A}\xi_1]$ and $P^\perp$ is zero on $[\mathscr{A}\xi_1]$). Since $\mathscr{A}$ has trivial null space we conclude $P^\perp\xi_1 = 0$. Finally, since T must commute with $P \in \mathscr{A}'$ it must leave the range of P invariant, and thus $T\xi_1 \in$ range $P = [\mathscr{A}\xi_1]$. This means we can find $S \in \mathscr{A}$ such that $\|T\xi_1 - S\xi_1\| < \varepsilon$, as required.

Now the case of general $n \geqslant 2$ is reduced to the above by the following device. Fix n, and let $\mathscr{H}_n = \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ be the direct sum of n copies of the underlying Hilbert space $\mathscr{H}$. Choose $\xi_1, \ldots, \xi_n \in \mathscr{H}$ and define $\eta \in \mathscr{H}_n$ by $\eta = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$. Let $\mathscr{A}_n \subseteq \mathscr{L}(\mathscr{H}_n)$ be the $*$-algebra of all operators of the form $\{S \oplus S \oplus \cdots \oplus S : S \in \mathscr{A}\}$. Thus each element of $\mathscr{A}_n$ can be expressed as a diagonal $n \times n$ operator matrix

$$\begin{pmatrix} S & & & 0 \\ & S & & \\ & & \ddots & \\ 0 & & & S \end{pmatrix}$$

$S \in \mathscr{A}$. The reader can see by a straightforward calculation that an $n \times n$ operator matrix (T_{ij}), $T_{ij} \in \mathscr{L}(\mathscr{H})$, commutes with $\mathscr{A}_n$ iff each entry T_{ij} belongs to $\mathscr{A}'$. This gives a representation for $\mathscr{A}'_n$ as operator matrices, and now a similar calculation shows that (T_{ij}) commutes with $\mathscr{A}'_n$ iff (T_{ij}) has the form

$$\begin{pmatrix} T & & & 0 \\ & T & & \\ & & \ddots & \\ 0 & & & T \end{pmatrix}$$

with $T \in \mathscr{A}''$. Thus we have a representation for $\mathscr{A}''_n$. Now choose $T \in \mathscr{A}''$ and let $T_n = T \oplus T \oplus \cdots \oplus T$. Then $T_n \in \mathscr{A}''_n$ so that the argument already given shows that $T_n\eta \in [\mathscr{A}_n\eta]$, thus we can find $S \in \mathscr{A}$ such that $S_n\eta$ is within ε of $T_n\eta$ in the norm of $\mathscr{H}_n$. In other words, $\sum_{k=1}^n \|T\xi_k - S\xi_k\|^2 < \varepsilon^2$, as required. □

By definition, a *von Neumann algebra* is a self-adjoint subalgebra $\mathscr{R}$ of $\mathscr{L}(\mathscr{H})$ which contains the identity and is closed in the weak operator topology. Note that 1.2.1. asserts that such an $\mathscr{R}$ satisfies $\mathscr{R} = \mathscr{R}''$, and this gives a convenient criterion for an operator T to belong to $\mathscr{R}$: one simply checks to see if T commutes with $\mathscr{R}'$. As an illustration of this, let us consider the polar decomposition. That is, let $T \in \mathscr{L}(\mathscr{H})$, and let $|T|$ denote the positive square root of the positive operator T^*T (via the functional calculus). Then $|T| \in C^*(T^*T)$, and in particular $|T|$ belongs to the von Neumann algebra generated by T. We want to define a certain operator U such that $U|T| = T$. Note first that for all $\xi = \mathscr{H}$ we have $\||T|\xi\|^2 = (|T|\xi, |T|\xi) = (|T|^2\xi, \xi) = (T^*T\xi, \xi) = (T\xi, T\xi) = \|T\xi\|^2$. Therefore the map $U: |T|\xi \to T\xi$, $\xi \in \mathscr{H}$, extends uniquely to a linear isometry of the closed range of $|T|$ onto the closed range of T. Extend U to a bounded operator on $\mathscr{H}$ by putting $U = 0$ on the orthogonal complement of $|T|\mathscr{H}$. Then U is a partial isometry (i.e., U^*U is a projection) whose initial space is $[|T|\mathscr{H}]$ and which satisfies $U|T| = T$. It is easy to see that these properties determine U uniquely, and the above formula relating U and $|T|$ to T is called the *polar decomposition* of T. Now we want to show that U belongs to the von Neumann algebra generated by T. By 1.2.1, it suffices to show that U commutes with every operator Z which commutes with both T and T^*. Now in particular Z commutes with the self-adjoint operator $|T|$, and therefore Z leaves both $|T|\mathscr{H}$ and $(|T|\mathscr{H})^\perp$ invariant. In particular Z leaves the null space of $U(=(|T|\mathscr{H})^\perp)$ invariant and so $ZU = UZ = 0$ on the null space of U. Thus it suffices to show that $ZU = UZ$ on every vector of the form $|T|\xi$, $\xi \in \mathscr{H}$. But $ZU|T|\xi = ZT\xi = TZ\xi$, while $UZ|T|\xi = U|T|Z\xi = TZ\xi$, and we are done. This proves the following

Corollary. *Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$. Then both factors U and $|T|$ belong to the von Neumann algebra generated by T.*

The following density theorem is a special case of a theorem of Kaplansky [16]. For a set of operators $\mathscr{S}$ we will write ball $\mathscr{S}$ for the closed unit ball in $\mathscr{S}$, ball $\mathscr{S} = \{S \in \mathscr{S}: \|S\| \leq 1\}$.

Theorem 1.2.2. *Let $\mathscr{A}$ be a self-adjoint algebra of operators and let $\mathscr{A}_s$ be the closure of $\mathscr{A}$ in the strong operator topology. Then every self-adjoint element in* ball *$\mathscr{A}_s$ can be strongly approximated by self-adjoint elements in* ball *$\mathscr{A}$.*

Proof. Note first that every self-adjoint element in the unit ball of the *norm* closure of $\mathscr{A}$ can be norm-approximated by self-adjoint elements in ball $\mathscr{A}$. Thus we can assume $\mathscr{A}$ is norm closed.

Now since the $*$-operation is not strongly continuous, we cannot immediately assert that the strong closure of the convex set $\mathscr{S}$ of self-adjoint elements of $\mathscr{A}$ contains $\{T \in \mathscr{A}_s: T = T^*\}$. But its *weak* closure does (because if a net S_n converges to $T = T^*$ strongly, then the real parts of S_n converge weakly to T), and moreover since the weak and strong operator topologies have the same continuous linear functionals (Exercise 1.2.E) they must also have the same closed convex sets. Thus we see in this way that the strong closure of $\mathscr{S}$ contains the self-adjoint elements of $\mathscr{A}_s$.

Now consider the continuous functions $f: \mathbb{R} \to [-1, +1]$ and $g: [-1, +1] \to \mathbb{R}$ defined by $f(x) = 2x(1 + x^2)^{-1}$ and $g(y) = y(1 + \sqrt{1 - y^2})^{-1}$. Then we have $f \circ g(y) = y$, for all $y \in [-1, +1]$, and clearly $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. We claim that the map $S \mapsto f(S)$ is strongly continuous on the set of all self-adjoint operators on $\mathscr{H}$. Granting that for a moment, note that 1.2.2 follows. For if $T = T^* \in \mathscr{A}_s$ is such that $\|T\| \leqslant 1$, then $S_0 = g(T)$ is a self-adjoint element of $\mathscr{A}_s$, so that by the preceding paragraph there is a net S_n of self-adjoint elements of $\mathscr{A}$ which converges strongly to S_0. Hence $f(S_n) \to f(S_0)$ strongly. Now $f(S_n)$ is self-adjoint, belongs to $\mathscr{A}$ (because $\mathscr{A}$ is norm closed), and has norm $\leqslant 1$ since $|f| \leqslant 1$ on $\mathbb{R}$. On the other hand, $f \circ g(y) = y$ on $[-1, 1]$ implies $f(S_0) = f \circ g(T) = T$, because $\|T\| \leqslant 1$, and this proves T is the strong limit of self-adjoint elements of ball $\mathscr{A}$.

Finally, the fact that $f(S) = 2S(I + S^2)^{-1}$ is strongly continuous follows after a moments reflection upon the operator identity

$$2[f(S) - f(S_0)] = 4(1 + S^2)^{-1}(S - S_0)(1 + S_0^2)^{-1} - f(S)(S - S_0)f(S_0),$$

considering S tending strongly to S_0 □

Kaplansky also proved that ball $\mathscr{A}$ is strongly dense in ball $\mathscr{A}_s$. That is not obvious from what we have said, but a simple trick using 2×2 operator matrices allows one to deduce that from 1.2.2 (Exercise 1.2.D).

Corollary. *Let $\mathscr{A}$ be a self-adjoint algebra of operators on a separable Hilbert space $\mathscr{H}$. Then for every operator T in the strong closure of $\mathscr{A}$, there is a sequence $T_n \in \mathscr{A}$ such that $T_n \to T$ in the strong operator topology.*

Proof. We can assume $\|T\| \leqslant 1$, and since we can argue separately with the real and imaginary parts of T, we can assume $T = T^*$. Let $\xi_1, \xi_2, \ldots$ be a countable dense set in $\mathscr{H}$. By 1.2.2, for each $n \geqslant 1$, we can find a self-adjoint element T_n in $\mathscr{A}$ such that $\|T_n\| \leqslant 1$ and $\|T_n\xi_k - T\xi_k\| < 1/n$ for $k = 1, 2, \ldots, n$. Thus $T_n \to T$ strongly on the dense set $\{\xi_1, \xi_2, \ldots\}$ of $\mathscr{H}$, and since $\|T_n\| \leqslant 1$, the corollary follows. □

This corollary shows that in the separable case, the strong closure of a C^*-algebra of operators can be achieved by adjoining to the algebra all limits of its strongly convergent sequences.

A C^*-algebra is *separable* if it has countable norm-dense subset. A separable C^*-algebra is obviously countably generated (a countable dense set clearly generates), and the reader can easily verify the converse: every countably generated C^*-algebra is separable. We conclude this section by pointing out a useful relation between separably-acting von Neumann algebras and separable C^*-algebras.

Let $\mathscr{H}$ be a Hilbert space. Then it is well known that the closed unit ball in $\mathscr{L}(\mathscr{H})$ is compact in the relative weak operator topology ([7], p. 34). Moreover, note that if $\mathscr{H}$ is separable then ball $\mathscr{L}(\mathscr{H})$ is a compact *metric space*. Indeed, if $u_1, u_2, \ldots$ is a countable dense set of unit vectors in $\mathscr{H}$ then the function

$$d(S, T) = \sum_{i,j=1}^{\infty} 2^{-i-j}|(Su_i - Tu_i, u_j)|$$