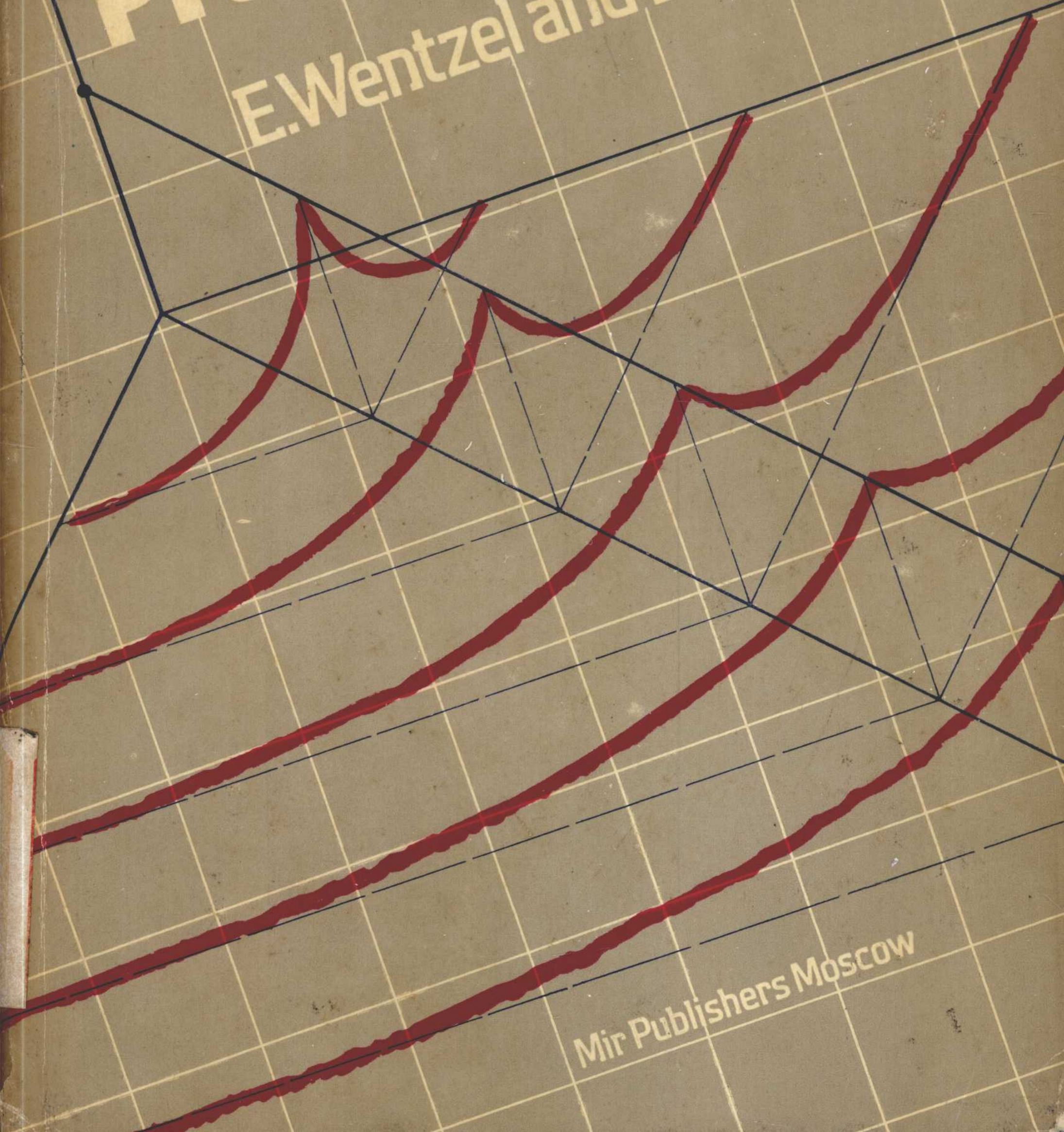
Applied Problems in Probability Theory

E. Wentzel and L. Ovcharov



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CONTENTS

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Е. С. Вентцель, Л. А Овчаров

Прикладные задачи по теории вероятностей

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AUTHORS' PREFACE

This book is based on many years of experience of teaching probability theory and its applications at higher educational establishments. It contains many of the problems we ourselves encountered in our research and consultative work. The problems are related to a variety of fields including electrical engineering, radio engineering, data transmission, computers, information systems, reliability of technical devices, preventive maintenance and repair, accuracy of apparatus, consumer service, transport, and the health service.

The text is divided into eleven chapters; each of which begins with a short theoretical introduction which is followed by relevant formulas.

The problems differ both in the fields of application and in difficulty. At the beginning of each chapter the reader will find comparatively simple problems whose purpose is to help the reader grasp the fundamental concepts and acquire and consolidate the experience of applying probabilistic methods. Then follow more complicated applied problems, which can be solved only after the requisite theoretical knowledge has been acquired and the necessary techniques mastered.

We have avoided the standard typical problems which can be solved mechanically. Many problems may prove difficult for both beginners and experienced readers alike (the problems we believe most difficult are marked by an asterisk). In the interest of the reader most of the problems have both answers and detailed solutions, and they are given immediately after the problem rather than at the end of the book; we wrote the book for a laborious and thoughful reader who will first try to find his own answer. This structure is very convenient and has justified itself in another book we have written, *The Theory of Probability* (Nauka, Moscow, 1973), which has been republished many times both at home and abroad (some problems in that edition have been repeated in this book).

We believe that statements and detailed solutions of nontrivial problems which demonstrate certain of the techniques of problem solving are particularly interesting. Our aim is not just to solve a problem but to use the simplest and most general technique. Some problems have been given several different solutions. In many cases a method of solution used has a general nature and can be applied in several fields. We have paid special attention to numerical characteristics which makes it possible to solve a number of problems with exceptional simplicity. The applied problems using the theory of Markov stochastic processes has been given the greatest consideration.

The brief theoretical sections which open each chapter do not usually repeat what existing textbooks on probability theory present, but have been based on new methods.

Thus this book is, in a certain sense, intermediate between a collection of problems and a textbook on the theory of probabilities. It should be useful to a wide variety of readers such as students and lecturers at higher schools, engineers and research workers who require a probabilistic approach to their work. Note that the detailed solutions and the consideration given to problem solving techniques make this book suitable for independent study.

We wish to express our gratitude to B. V. Gnedenko, Academician of the Ukrainian Academy of Sciences, who read the manuscript very attentively and made a number of valuable remarks, and also to V. S. Pugachev, Academician of the USSR Academy of Sciences, whom we consulted frequently when we worked on the book and whose methodical principles and notation we substantially followed.

PREFACE TO THE ENGLISH EDITION

We wish to express our satisfaction at having the opportunity to bring our techniques of solving applied problems in probability theory to the notice of the English reader.

We wrote this book so that it could be used both as a study aid in probability theory and as a collection of problems, of which are about 700.

The theoretical part at the beginning of each chapter and the methodical instructions for solving the various problems make it possible to use the book as a study aid. The solutions of many of the problems in the book are important both from an educational point of view and because they can be used when investigating various applied engineering problems.

The methodology and the notation in this book correspond, in the main, to those used in V. S. Pugachev's book [6] which was recently published in Great Britain.

When the book was being prepared for translation into English, a number of corrections and additions were made which improved the content of the book. During this preparatory work, Assistant Professor Danilov made an essential contribution, for which we want to express our gratitude.

E. Wentzel, L. Ovcharov

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Fundamental Concepts of Probability Theory. Direct Calculation of Probability in an Urn Model

1.0. The theory of probability is the mathematical study of random phenomena. The concept of a *random event* (or simply event) is one of the principal concepts in probability theory. An *event* is the outcome of an experiment (or a trial). Six dots appearing on the top face of a die, the failure of a device during its service life, and a distortion in a message transmitted over a communication channel are all examples of events. Each event has an associated *quantity* which characterizes how likely its occurrence is; this is called the *probability of the event.*

There are several approaches to the concept of probability. The "classical" approach is to calculate the number of favourable outcomes of a trial and divide it by the total number of possible outcomes [see formula (1.0.6) below]. The frequency or statistical approach is based on the concept of the frequency of an event in a long series of trials. The *frequency* of an event in a series of N trials is the ratio of the number of trials in which it occurs to the total number of trials. There are random events for which a *stability of the frequencies* is observed; with an increase in the number N of independent trials, the frequency of the event stabilizes, and tends to a certain constant quantity, which is called the *probability of an event.*

The modern construction of probability theory is based on an axiomatic approach using the fundamental concepts of set theory. This approach to probability theory is known as the set-theoretical approach.

Here are the fundamental concepts of set theory.

A *set* is a collection of objects each of which is called an *element of the set.* A set of students who study at a given school, the set of natural numbers which do not exceed 100, a set of points on a plane lying within or on a circle with a unit radius and centre at the origin are all examples of sets.

There are several ways of designating sets. It can be denoted either by one capital letter or by the enumeration of its elements given in braces or by indicating (also in braces) a rule which associates an element with a set. For example, the set M of natural numbers from 1 to 100 can be written

$$M = \{1, 2, \ldots, 100\} = \{i \text{ is an integer; } 1 \leqslant i \leqslant 100\}.$$

The set C of points on a plane which lie within or on a circle with centre at the origin can be written in the form $C = \{x^2 + y^2 \leqslant R^2\}$, where x and y are the Cartesian coordinates of the point and R is the radius of the circle.

Depending on how many elements it has, a set may be *finite* or *infinite*. The set $M = \{1, 2, \ldots, 100\}$ is finite and contains 100 elements (in a particular case a finite set can consist of only one element). The set of all natural numbers $N = \{1, 2, \ldots, n, \ldots\}$ is infinite; the set of even numbers $N_2=\{2, 4, \ldots, 2n, \ldots\}$ is also infinite. An infinite set is said to be *countable* if all of its terms can be enumerated (both of the infinite sets N and N_2 given above are countable). The set C of points within or on a circle of radius $R > 0$

$$C = \{x^2 + y^2 \leqslant R\} \tag{1.0.1}$$

is infinite and uncountable (its elements cannot be enumerated).

Two sets A and B *coincide* (or are equivalent) if they consist of the same elements (the coincidence of sets is expressed thus: $A = B$). For example, the set of roots of the equation $x^2 - 5x + 4 = 0$ coincides with the set $\{1, 4\}$ (and also with the set $\{4, 1\}$).

The notation $a \in A$ means that an object a is an element of a set A; or, in other words, *a belongs to A*. The notation $a \notin A$ means that an object a is not an element of a set A.

An *empty set* is a set with no elements. It is designated $\varnothing$. Example: the set of points on a plane whose coordinates (x, y) satisfy the inequality $x^2 + y^2 \leqslant -1$ is an empty set: $\{x^2 + y^2 \leqslant -1\} = \varnothing$. All empty sets are equivalent.

A set B is said to be a *subset* of a set A if all the elements of B also belong to A. The notation is $B \subseteq A$ (or $A \supseteq B$). Examples: $\{1, 2, \ldots, 100\} \subseteq \{1, 2, \ldots, 1000\}$; $\{1, 2, \ldots, 100\} \subseteq \{1, 2, \ldots, 100\}$; $\{x^2 + y^2 \leqslant 1\} \subseteq \{x^2 + y^2 \leqslant 2\}$.

An empty set is a subset of any set A: $\varnothing \subseteq$ A.

We can use a geometrical interpretation to depict the inclusion of sets; the points on the plane are elements of the set (see Fig. 1.0.1, where B is a subset of A).

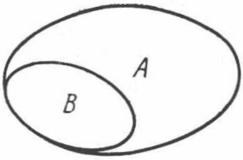

Fig. 1.0.1

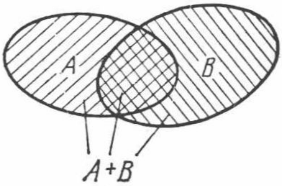

Fig. 1.0.2

The *union* (logical sum) of the sets A and B is a set $C = A + B$ which consists of all the elements of A and all those of B (including those which belong to both A and B). In short, a union of two sets is a collection of elements belonging to *at least one* of them. Examples: $\{1, 2, \ldots, 100\} + \{50, 51, \ldots, 200\} = \{1, 2, \ldots, 200\}$, $\{1, 2, \ldots 100\} + \{1, 2, \ldots, 1000\} = \{1, 2, \ldots, 1000\}$, $\{1, 2, \ldots, 100\} + \varnothing = \{1, 2, \ldots, 100\}$. The union of the two sets A and B is shown in Fig. 1.0.2; the shaded area is $A + B$.

We can similarly define a union of any number of sets: $A_1 + A_2 + \ldots + A_n = \sum_{i=1}^{n} A_i$ is a set consisting of all the elements which belong to at least one of the sets $A_1, \ldots, A_n$. We can also consider a union of an infinite (countable) number of sets $\sum_{i=1}^{\infty} A_i = A_1 + A_2 + \ldots + A_n + \ldots$. Example: $\{1, 2\} + \{2, 3\} + \{3, 4\} + \ldots + \{n, n-1\} + \ldots = \{1, 2, 3, \ldots, n, \ldots\}$.

The *intersection* (logical product) of two sets A and B is the set $D = A \cdot B$ that consists of the elements which belong to both A and B. Examples: $\{1, 2, \ldots, 100\} \times \{50, 51, \ldots, 200\} = \{50, 51, \ldots, 100\}$, $\{1, 2, \ldots 100\} \cdot \{1, 2, \ldots, 1000\} = \{1, 2, \ldots, 100\}$, $\{1, 2, \ldots, 100\} \cdot \varnothing = \varnothing$. An intersection of two sets A and B is shown in Fig. 1.0.3.

We can similarly define the intersection of any number of sets. The set $A_1 \cdot A_2 \ldots A_n = \prod_{i=1}^{n} A_i$ consists of all the elements which belong to all the sets $A_1, A_2, \ldots A_n$ simultaneously. The definition can be extended to an infinite (countable) number of sets: $\prod_{i=1}^{\infty} A_i$ is a set consisting of elements belonging to all the sets $A_1, A_2, \ldots, A_n, \ldots$ simultaneously.

Two sets A and B are said to be *disjoint* (nonintersecting) if their intersection is an empty set: $A \cdot B = \varnothing$, i.e. no element belongs to both A and B (Fig. 1.0.4). Figure 1.0.5 illustrates several disjoint sets.

As in the notation of an ordinary multiplication, the $\cdot$ sign is usually omitted.

It is sufficient to have this elementary knowledge of set theory in order to use the set-theoretical construction of probability theory.

Assume that an experiment (trial) is conducted whose result is not known beforehand, i.e. is accidental. Let us consider the set Ω of all possible outcomes of the experiment: each of its elements $\omega \in \Omega$ (each outcome) is known as an *elementary event* and the whole set Ω as the *space of elementary events* or the *sample space*. Any subset of the set Ω is known as an *event* (or *random event*); and any event A is a subset of the set Ω, viz. $A \subseteq \Omega$.

Example: an experiment involves tossing a die; the space of elementary events $\Omega = \{1, 2, 3, 4, 5, 6\}$; an event A is an even score; $A = \{2, 4, 6\}$; $A \subseteq \Omega$. In particular, we can consider the event Ω (since every set is a subset of itself); it is said to be a *certain event* (it must occur in every experiment). We can add the empty set $\varnothing$ to the whole space Ω of elementary events; this set is also an event and is said to be an *impossible event* (it cannot occur as a result of an experiment). An example

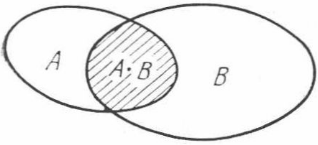

Fig. 1.0.3

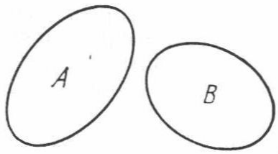

Fig. 1.0.4

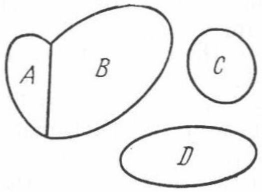

Fig. 1.0.5

of a certain event: {a score not exceeding 6 when a die is tossed}; an example of an impossible event: {a score of 7 dots on a face of a die}.

Note that there are different ways of defining an elementary event in the same experiment; say, in a random throw of a point on a plane, the position of the point can be defined both by a pair of Cartesian coordinates (x, y) and by a pair of polar coordinates (ρ, φ).

Two mutually disjoint events A and B (such that $AB = \varnothing$) are said to be *incompatible*; the occurrence of one precludes the occurrence of the other. Several events $A_1, A_2, \ldots, A_n$ are said to be *pairwise incompatible* (or simply incompatible), or *two-by-two mutually exclusive events*, if the occurrence of one of them precludes the occurrence of each of the others.

We say that several events $A_1, A_2, \ldots, A_n$ *form a complete group* if $\sum_{i=1}^{n} A_i = \Omega$, i.e. if their sum is a certain event (in other words, if at least one of them is certain to occur as a result of an experiment). Example: an experiment consists in tossing a die, the events $A = \{1, 2\}$, $B = \{2, 3, 4\}$, and $C = \{4, 5, 6\}$ form a complete group; $A + B + C = \{1, 2, 3, 4, 5, 6\} = \Omega$.

We now introduce axioms of probability theory. Assume that every event is associated with a *number* called its *probability*. The probability of an event A is designated as $P(A)$*). We require that the probabilities of the events should satisfy the following axioms.

I. *The probability of an event A falls between zero and unity*

$$0 \leqslant P(A) \leqslant 1. \tag{1.0.2}$$

II. Probability addition rule: *if A and B are mutually exclusive, then*

$$P(A + B) = P(A) + P(B). \tag{1.0.3}$$

Axiom (1.0.3) can be immediately generalized to any finite number of events: *if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then*

$$P\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i). \tag{1.0.4}$$

---

*) If an event (a set) is denoted by a verbal description of its properties, or by a formula of type (1.0.1), or by an enumeration of the elements of the set rather than by a letter, we do not use parentheses but use braces when designating the probability, e.g. $P\{x^2 + y^2 < 2\}$.

III. Probability addition rule for an infinite sequence of events: *if* $A_1, A_2, \ldots, A_n$ *are mutually exclusive events, then*

$$P\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{1.0.5}$$

The axioms of probability theory can be used to calculate the probability of any events (the subsets of Ω) from the probabilities of the elementary events (if there is a finite or countable number of them). It is not necessary to consider here the ways of determining the probability of the elementary events. In practice they are found either from a consideration of the symmetry of the experiment (for a symmetric die, for instance, it is natural to assume the appearance of each face to be equipossible), or on the basis of experimental data (frequencies).

If the possible outcomes of an experiment have symmetry, then the probabilities can be directly calculated from the so-called urn model (*model of events*). This technique is based on the assumption that the elementary events are equipossible. Several events $A_1, A_2, \ldots, A_n$ are said to be *equipossible* if they have the same probability by virtue of the symmetry of the conditions of the experiment relative to those events: $P(A_1) = P(A_2) = \ldots = P(A_n)$.

If, in an experiment, we can represent the sample space Ω as a complete group of disjoint and equipossible events $\omega_1, \omega_2, \ldots, \omega_n$, then the events are called *cases* (*chances*) and the experiment is said to reduce to the urn model.

A case ω_i is said to be *favourable* to an event A if it is an element of the set A: $\omega_i \in A$.

Since the cases $\omega_1, \omega_2, \ldots, \omega_n$ form a complete group of events, it follows that

$$\sum_{i=1}^{n} \omega_i = \Omega.$$

Since the elementary events $\omega_1, \omega_2, \ldots, \omega_n$ are incompatible, it follows, from the probability addition rule, that

$$P\left(\sum_{i=1}^{n} \omega_i\right) = P(\Omega) = \sum_{i=1}^{n} P(\omega_i) = 1.$$

Since the elementary events $\omega_1, \omega_2, \ldots, \omega_n$ are equipossible, their probability is the same and is equal to $1/n$:

$$P(\omega_1) = P(\omega_2) = \ldots = P(\omega_n) = 1/n.$$

This formula yields a so-called *classical formula* for the probability of an event: *if an experiment reduces to an urn model, then the probability of event A in that experiment can be calculated by the formula*

$$P_A(A) = m_A/n, \tag{1.0.6}$$

where m_A is the number of cases favourable to the event A, and n is the total number of cases.

Formula (1.0.6), which was once accepted as the definition of probability, is now, with the modern axiomatic approach, a corollary of the probability addition rule.

Let us consider an example. Three white and four black balls are thoroughly stirred in an urn and a ball is drawn at random. Construct the sample space for this experiment and find the probability of the event A = {a white ball is drawn}. We now label the balls 1 to 7 inclusive; the first three balls are white and the last four are black. Hence

$$\Omega = \{1, 2, 3, 4, 5, 6, 7\}; \quad A = \{1, 2, 3\}.$$

Since the experimental conditions are symmetric with respect to all the balls (a ball is drawn at random), the elementary events are equipossible. Since they are incompatible and form a complete group, the probability of event A can be found by formula (1.0.6): $P(A) = m_A/n = 3/7$.

When the experiment is symmetric with respect to the possibility of outcomes, formula (1.0.6) makes it possible to calculate the probabilities of events directly from the conditions of the experiment.

A "geometrical" approach to the calculation of the probabilities of events is a natural generalization and extension of a direct calculation of probabilities in the urn model. It is employed when the sample space Ω includes an uncountable set of elementary events $\omega \in \Omega$ and by symmetry none of them is more likely than the

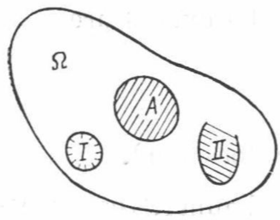

Fig. 1.0.6

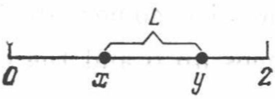

Fig. 1.0.7

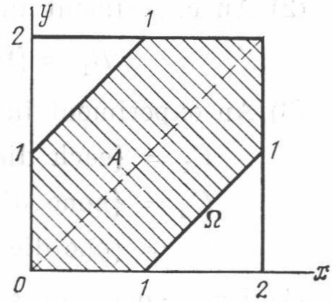

Fig. 1.0.8

others as concerns the possibility of occurrence*). Assume that the sample space Ω is a domain on a plane (Fig. 1.0.6) and that the elementary events ω are points within the domain. If the experiment has a symmetry of possible outcomes (say, a "point" object is thrown at random in the interior of the domain), then all the elementary events are "equal in rights" and it is natural to assume that the probabilities that the elementary event ω will fall in domains I and II of the same size S are equal and the probability of any event $A \subseteq \Omega$ is equal to the ratio of the area S_A of domain A to the area of the whole domain Ω:

$$P(A) = S_A/S_\Omega. \tag{1.0.7}$$

Formula (1.0.7) is a generalization of the classical formula (1.0.6) to an uncountable set of elementary events. The symmetry of the experimental conditions with respect to its elementary outcomes ω is usually formulated using the words "at random". In essence this is equivalent to the random choice of a ball in an urn (see above). In textbooks the probabilities calculated by this technique are often called "geometrical probabilities".

Assume, for instance, that two points with abscissas x and y are put at random on the interval from 0 to 2 (Fig. 1.0.7). Find the probability that the distance L between them is less than unity. The elementary event ω is characterized by a pair of coordinates (x, y). The space of elementary events is a square with side 2 on the x, y plane (Fig. 1.0.8). We then have $L = |y - x|$ and so the event $A = \{|y - x| < 1\}$ is associated with the domain A which is hatched in Fig. 1.0.8.

$$P(A) = P\{|y - x| < 1\} = S_A/S_\Omega = 3/4.$$

If the space of elementary events is not plane but three-dimensional, then we must replace the areas S_A and S_Ω in formula (1.0.7) by the volumes V_A and V_Ω, and for a one-dimensional space, by the lengths L_A and L_Ω of the corresponding segments of a straight line.

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*) We do not say that the elementary events ω are "equipossible"; we shall ascertain in Chapter 5 that the probability of each of them is equal to zero.

Problems and Exercises

1.1. Find out whether the events indicated in each example form a complete group of events for the given experiment (answer yes or no).

(1) An experiment involves tossing a coin; the events are

$$A_1 = \{\text{heads}\};\ A_2 = \{\text{tails}\}.$$

(2) An experiment involves tossing two coins; the events are

$$B_1 = \{\text{two heads}\};\ B_2 = \{\text{two tails}\}.$$

(3) An experiment involves throwing two dice; the events are

$$C_1 = \{\text{each die comes up } 6\}:$$
$$C_2 = \{\text{none of the dice comes up } 6\};$$
$$C_3 = \{\text{one die comes up 6 and the other does not}\}.$$

(4) Two signals are sent over a communication channel; the events are

$$D_1 = \{\text{at least one signal is not distorted}\};$$
$$D_2 = \{\text{at least one signal is distorted}\}.$$

(5) Three messages are sent over a communication channel; the events are

$E_1 = \{$the three messages are transmitted without an error$\}$;

$E_2 = \{$the three messages are transmitted with errors$\}$;

$E_3 = \{$two messages are transmitted with errors and one without an error$\}$.

Answer. (1) yes, (2) no, (3) yes, (4) yes, (5) no.

1.2. Regarding each group of events say whether they are incompatible in the given experiment (yes, no).

(1) An experiment involves tossing a coin; the events are

$$A_1 = \{\text{a head}\};\ A_2 = \{\text{a tail}\}.$$

(2) An experiment involves tossing two coins; the events are

$$B_1 = \{\text{the first coin comes up heads}\};$$
$$B_2 = \{\text{the second coin comes up heads}\}.$$

(3) Two shots are fired at a target; the events are

$$C_0 = \{\text{no hits}\};\ C_1 = \{\text{one hit}\};\ C_2 = \{\text{two hits}\}.$$

(4) The same as above; the events are

$$D_1 = \{\text{one hit}\};\quad D_2 = \{\text{one miss}\}.$$

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