THIRD EDITION

A Transition to Advanced Mathematics

Douglas Smith

Maurice Eggen

Richard St. Andre

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Douglas Smith
University of North Carolina at Wilmington

Maurice Eggen
Trinity University

Richard St. Andre
Central Michigan University

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PREFACE

To the First Edition

"I understand mathematics but I just can't do proofs."

Our experience has led us to believe that the remark above, though contradictory, expresses the frustration many students feel as they pass from beginning calculus to a more rigorous level of mathematics. This book developed from a series of lecture notes for a course at Central Michigan University that was designed to address this lament. The text is intended to bridge the gap between calculus and advanced courses in at least three ways. First, it provides a firm foundation in the major ideas needed for continued work. Second, it guides students to think and to express themselves mathematically—to analyze a situation, extract pertinent facts, and draw appropriate conclusions. Finally, we present introductions to modern algebra and analysis of sufficient depth to capture some of their spirit and characteristics.

We begin in Chapter 1 with a study of the logic required by mathematical arguments, discussing not formal logic but rather the standard methods of mathematical proof and their validity. Methods of proof are examined in detail, and examples of each method are analyzed carefully. Denials are given special attention, particularly those involving quantifiers. Techniques of proof given in this chapter are used and referred to later in the text. Although the chapter was written with the idea that it may be assigned as out-of-class reading, we find that most students benefit from a thorough study of logic.

Much of the material in Chapters 2, 3, and 4 on sets, relations, and functions, will be familiar to the student. Thus, the emphasis is on enhancing the student's ability to write and understand proofs. The pace is deliberate. The rigorous approach requires the student to deal precisely with these concepts.

Chapters 5, 6, and 7 make use of the skills and techniques the student has acquired in Chapters 1 through 4. These last three chapters are a cut above the earlier chapters in terms of level and rigor. *Chapters* 1 *through* 4 *and any one of Chapters* 5, 6, *or* 7 *provide sufficient material for a one-semester course.* An alternative is to choose among topics by selecting, for example, the first two sections of Chapter 5, the first three sections of Chapter 6, and the first two sections of Chapter 7.

Chapter 5 begins the study of cardinality by examining the properties of finite and infinite sets and establishing countability or uncountability for the familiar number systems. The emphasis is on a working knowledge of cardinality—particularly countable sets, the ordering of cardinal numbers, and applications of the Cantor–Schröeder–Bernstein Theorem. We include a brief discussion of the Axiom of Choice and relate it to the comparability of cardinals.

Chapter 6, which introduces modern algebra, concentrates on the concept of a group and culminates in the Fundamental Theorem of Group Homomorphisms. The idea of an operation preserving map is introduced early and developed throughout the section. Permutation groups, cyclic groups, and modular arithmetic are among the examples of groups presented.

Chapter 7 begins with a description of the real numbers as a complete ordered field. We continue with the Heine–Borel Theorem, the Bolzano–Weierstrass Theorem, and the Bounded Monotone Sequence Theorem (each for the real number system), and then return to the concept of completeness.

Exercises marked with a solid star have complete answers at the back of the text. Open stars indicate that a hint or a partial answer is provided. "Proofs to Grade" are a special feature of most of the exercise sets. We present a list of claims with alleged proofs, and the student is asked to assign a letter grade to each "proof" and to justify the grade assigned. Spurious proofs are usually built around a single type of error, which may involve a mistake in logic, a common misunderstanding of the concepts being studied, or an incorrect symbolic argument. Correct proofs may be straightforward, or they may present novel or alternate approaches. We have found these exercises valuable because they reemphasize the theorems and counterexamples in the text and also provide the student with an experience similar to grading papers. Thus, the student becomes aware of the variety of possible errors and develops the ability to read proofs critically.

In summary, our main goals in this text are to improve the student's ability to think and write in a mature mathematical fashion and to provide a solid understanding of the material most useful for advanced courses. Student readers, take comfort from the fact that we do not aim to turn you into theorem-proving wizards. Few of you will become research mathematicians. Nevertheless, in almost any mathematically related work you may do, the kind of reasoning you need to be able to do is the same reasoning you use in proving theorems. You must first understand exactly what you want to prove (verify, show, or explain), and you must be familiar with the logical steps that allow you to get from the hypothesis to the conclusion. Moreover, a proof is the ultimate test of your understanding of the subject matter and of mathematical reasoning.

We are grateful to the many students who endured earlier versions of the manuscript and gleefully pointed out misprints. We acknowledge also the helpful comments of Edwin H. Kaufman, Melvin Nyman, Mary R. Wardrop, and especially Douglas W. Nance, who saw the need for a course of this kind at CMU and did a superb job of reviewing the manuscript.

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Third Edition

The intent of the second and third editions has been to rework and revise the text selectively while maintaining its character as an introduction both to the foundational topics of sets, relations, and functions and to the rigor of mathematical thinking and writing. Explanations, examples, and exercises have been added or revised throughout the text. There are extensive reworkings in Chapters 1 and 5. A new optional section 1.6 expands on examples of proof techniques. The optional section on graphs added to Chapter 3 in the second edition has been rewritten in the third.

We have resisted the addition of topics that would be "nice" but peripheral to the core material of the first four chapters. We continue to find that most instructors follow the development of the text, covering either Chapters 1–4, or 1–5, and use portions of the last independent chapters to introduce topics from algebra or analysis as time permits. Many instructors prefer to treat selectively the cardinality topics in Chapter 5. One common approach is to treat the definitions and results in the first two sections on finite and countable sets, the definition of cardinal number and Cantor's Theorem from the next section, and the facts about countable sets in the last section of Chapter 5.

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Richard St. Andre
Douglas Smith

CONTENTS

CHAPTER 1

Logic and Proofs

Although mathematics is both a science and an art, special characteristics distinguish mathematics from the humanities and from other sciences. Particularly important is the kind of reasoning that typifies mathematics. The natural or social scientist generally makes observations of particular cases or phenomena and seeks a general theory that describes or explains the observations. This approach is called **inductive reasoning**, and it is tested by making further observations. If the results are incompatible with theoretical expectations, the scientist usually must reject or modify the theory.

The mathematician, too, frequently uses inductive reasoning as he or she attempts to describe patterns and relationships among quantities and structures. The characteristic thinking of the mathematician, however, is **deductive reasoning**, in which one uses logic to draw conclusions based on statements accepted as true. The conclusions of a mathematician are proved to be true, *provided that the assumptions are true.* If the results of a mathematical theory are deemed incompatible with some portion of reality, the fault lies not in the theory but with the assumptions about reality that make the theory inapplicable to that portion of reality. Indeed, the mathematician is not restricted to the study of observable phenomena, even though one can trace the development of mathematics back to the need to describe spatial relations (geometry) and motion (calculus) or to solve financial problems (algebra). Using logic, the mathematician can draw conclusions about any mathematical structure imaginable.

The goal of this chapter is to provide a working knowledge of the basics of logic and the idea of proof, which are fundamental to deductive reasoning. This knowledge is important in many areas other than mathematics. For example, the thought processes used to construct an algorithm for a computer program are much like those used to develop the proof of a theorem.

1.1 Propositions and Connectives

Natural languages such as English allow for many types of sentences. Some sentences are interrogatory (Where is my sweater?), others exclamatory (Oh, no!), while others have a definite sense of truth to them (Abe Lincoln was the first U.S. president.). A **proposition** is a sentence that is either true or false. Thus a proposition has exactly one truth value: true which we denote by T, or false which we denote by F.

Some examples of propositions are:

(a) $\sqrt{2}$ is irrational.

(b) $1 + 1 = 5$.

(c) The elephant will become extinct on the planet Earth before the rhinoceros.

(d) Julius Caesar had two eggs for breakfast on his tenth birthday.

We are not concerned here with the difficulty of establishing the actual truth value of a proposition. We readily see that proposition (a) has the value T while (b) has the value F. It may take many years to determine whether proposition (c) is true or false, but its truth value will certainly be established if either animal ever becomes extinct. If both species (and the Earth) somehow survive forever, the statement is false. There may be no way ever to determine what value proposition (d) has. Nevertheless, each of the above is either true or false, hence is a proposition.

Here are some sentences that are not propositions:

(e) What did you say?

(f) $x^2 = 36$

(g) This sentence is false.

Sentence (e) is an interrogative statement that has no truth value. Sentence (f) could be true or false depending on what value x is assigned. We shall study sentences of this type in section 1.3.

Statement (g) is an example of a sentence that is neither true nor false, and is referred to as a **paradox**. If (g) is true, then by its meaning (g) must be false. On the other hand, if (g) is false, then what it purports is false, so (g) must be true. Thus, (g) can have neither T nor F for truth value. The study of paradoxes such as this has played a key role in the development of modern mathematical logic. A famous example of a paradox formulated by the English logician Bertrand Russell is discussed in section 2.1.

Propositions (a)–(d) are **simple** or **atomic** in the sense that they do not have any other propositions as components. **Compound** propositions can be formed by using logical connectives with simple propositions.

Definitions Given propositions P and Q,

The **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when *both* P and Q are true.

The **disjunction** of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when *at least one* of P or Q is true.

The **negation** of P, denoted $\sim P$, is the proposition "not P." $\sim P$ is true exactly when P is false.

If P is "$1 \neq 3$" and Q is "7 is odd," then

$P \wedge Q$ is "$1 \neq 3$ and 7 is odd."

$P \vee Q$ is "$1 \neq 3$ or 7 is odd."

$\sim Q$ is "It is not the case that 7 is odd."

Since in this example both P and Q are true, $P \wedge Q$ and $P \vee Q$ are true, while $\sim Q$ is false.

All of the following are true propositions:

"It is not the case that $\sqrt{10} > 4$."

"$\sqrt{2} < \sqrt{3}$ or chickens have lips."

"Venus is smaller than Earth or $1 + 4 = 5$."

"$6 < 7$ and $7 < 8$."

All of the following are false:

"1955 was a bad year for wine and π is rational."

"It is not the case that 10 is divisible by 2."

"$2^4 = 16$ and a quart is larger than a liter."

Other connectives commonly used in English are *but*, *while*, and *although*, each of which would normally be translated symbolically with the conjunction connective. A variant of the connective *or* is discussed in the exercises.

Example. Let M be "Milk contains calcium" and I be "Italy is a continent." Since M has the value T and I has the value F,

"Italy is a continent and milk contains calcium," symbolized $I \wedge M$, is false;

"Italy is a continent or milk contains calcium," $I \vee M$, is true;

"It is not the case that Italy is a continent," $\sim I$, is true.

An important distinction must be made between a proposition and the form of a proposition. In the previous example, "Italy is a continent and milk contains calcium" is a proposition with a single truth value (F), while the propositional form $P \wedge Q$, which may be used to symbolically represent the sentence, has no truth value in and of itself. The form $P \wedge Q$ is an expression that obtains a value T or F after specific propositions are designated for P and Q (when for instance, we let P be "Italy is a continent" and Q be "Milk contains calcium"), or when the symbols P and Q are given truth values.

By the form of a compound proposition we mean how the proposition is put together using logical connectives. For components P and Q, $P \wedge Q$ and $P \vee Q$ are two different propositional forms. Informally, a **propositional form** is an expression involving finitely many logical symbols (such as $\wedge$ and $\sim$) and letters. Expressions that are single letters or are formed correctly from the definitions of connectives are called **well-formed formulas**. For example, $(P \wedge (Q \vee \sim Q))$ is well formed, whereas $(P \vee Q \sim)$, $(\sim P \sim Q)$, and $\vee Q$ are not. A more precise definition and study of well-formed formulas may be found in Elliot Mendelson's *An Introduction to Mathematical Logic* (D. Van Nostrand, 1979).

The truth values of a compound propositional form are readily obtained by exhibiting all possible combinations of the truth values for its components in a truth table. Since each connective $\wedge$ and $\vee$ involves two components, their truth tables must list the four possible combinations of the truth values of those components. The truth tables for $P \wedge Q$ and $P \vee Q$ are

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Since the value of $\sim P$ depends only on the two possible values for P, its truth table is

P	$\sim P$
T	F
F	T

Frequently you will encounter compound propositions with more than two simple components. The propositional form $(P \wedge Q) \vee \sim R$ has three simple components; it follows that there are $2^3 = 8$ possible combinations of values for P, Q, R. The two main components are $P \wedge Q$ and $\sim R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee \sim R$
T	T	T	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	F	F	T	T
F	F	F	F	T	T

The propositional form $(\sim Q \vee P) \wedge (R \vee S)$ has 16 possible combinations of values for P, Q, R, S. Two main components are $\sim Q \vee P$ and $R \vee S$. Its truth table is shown here:

P	Q	R	S	$\sim Q$	$\sim Q \vee P$	$R \vee S$	$(\sim Q \vee P) \wedge (R \vee S)$
T	T	T	T	F	T	T	T
F	T	T	T	F	F	T	F
T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	T	T	T
F	T	F	T	F	F	T	F
T	F	F	T	T	T	T	T
F	F	F	T	T	T	T	T
T	T	T	F	F	T	T	T
F	T	T	F	F	F	T	F
T	F	T	F	T	T	T	T
F	F	T	F	T	T	T	T
T	T	F	F	F	T	F	F
F	T	F	F	F	F	F	F
T	F	F	F	T	T	F	F
F	F	F	F	T	T	F	F

Two propositions P and Q are **equivalent** if and only if they have the same truth value. The propositions "$1 + 1 = 2$" and "$6 < 10$" are equivalent (even though they have nothing to do with each other) because both are true. The ability to write equivalent statements from a given statement is an important skill in writing proofs. Of course, in a proof we expect some logical connection between such statements. This connection may be based on the form of the propositions.

Definition Two propositional forms are **equivalent** if and only if they have the same truth tables.

For example, the propositional forms $P \vee (Q \wedge P)$ and P are equivalent. To show this, we examine their truth tables.

P	Q	$Q \wedge P$	$P \vee (Q \wedge P)$
T	T	T	T
F	T	F	F
T	F	F	T
F	F	F	F

Since the P column and the $P \vee (Q \wedge P)$ column are identical, the propositional forms are equivalent. This means that, whatever propositions we choose to use for P and for Q, the results will be equivalent. If we let P be "91 is prime" and Q be "$1 + 1 = 2$," then "91 is prime" is equivalent to the proposition "91 is prime, or $1 + 1 = 2$ and 91 is prime." With these propositions for P and Q, Q is true and both P and $P \vee (Q \wedge P)$ are false. Thus, we have an instance of the second line of the truth table.

Any proposition P is equivalent to itself. Also the propositional forms P and $\sim(\sim P)$ are equivalent. Their tables are

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

> **Definition** A **denial** of a proposition S is any proposition equivalent to $\sim S$.

By definition, the negation $\sim P$ is a denial of the proposition P, but a denial need not be the negation. A proposition has only one negation but may have several denials. The ability to rewrite the negation of a proposition into a useful denial will be very important for writing indirect proofs (see section 1.4).

Example. The proposition P: "π is rational" has negation $\sim P$: "It is not the case that π is rational." Some useful denials are

"π is irrational."

"π is not the quotient of two integers."

"The decimal expansion of π is not repeating."

Note that since P is false, all denials of P are true.

Example. The proposition "The water is cold and the soap is not here" has two components, C: "The water is cold" and H: "The soap is here." The negation, $\sim(C \wedge \sim H)$, is "It is not the case that the water is cold and the soap is not here." Some other denials are

"Either the water is not cold or the soap is here."

"It is not the case that the water is cold and the soap is not here and the water is cold."

It may be verified by truth tables that the propositional forms $\sim C \vee H$ and $\sim[(C \wedge \sim H) \wedge C]$ are equivalent to $\sim(C \wedge \sim H)$.

Note that the negation in the last example is ambiguous when written in English. Does the "It is not the case" refer to the entire sentence or just to the component "The water is cold"? Ambiguities such as this are allowable in conversational English but can cause trouble in mathematics. To avoid ambiguities we introduce delimiters such as parentheses ( ), square brackets [ ], and braces { }. The negation above may be written symbolically as $\sim(C \wedge \sim H)$.

To avoid writing large numbers of parentheses, we use the rule that, first, $\sim$ applies to the smallest proposition following it, then $\wedge$ connects the smallest propositions surrounding it, and finally, $\vee$ connects the smallest propositions surrounding it. Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$. The negation of the disjunction $P \vee Q$ must be written with parentheses $\sim(P \vee Q)$. The propositional form $P \wedge \sim Q \vee R$ abbreviates $[P \wedge (\sim Q)] \vee R$. As further examples,

$P \vee Q \wedge R$ abbreviates $P \vee [Q \wedge R]$.

$P \wedge \sim Q \vee R$ abbreviates $[P \wedge (\sim Q)] \vee R$.

$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$.

When the same connective is used several times in succession, parentheses may also be omitted. We reinsert parentheses from the left, so that $P \vee Q \vee R$ is really $(P \vee Q) \vee R$. For example, $R \wedge P \wedge \sim P \wedge Q$ abbreviates $[(R \wedge P) \wedge (\sim P)] \wedge Q$, whereas $R \vee P \wedge \sim P \vee Q$, which does not use the same connective consecutively, abbreviates $(R \vee [P \wedge (\sim P)]) \vee Q$. Leaving out parentheses is not required; some propositional forms are easier to read with a few well-chosen "unnecessary" parentheses.

Some compound propositional forms always yield the value true just because of the nature of their form. **Tautologies** are propositional forms that are true for every assignment of truth values to their components. Thus a tautology will have the value true regardless of what proposition(s) we select for the components. For example, the Law of Excluded Middle, $P \vee \sim P$, is a tautology. Its truth table is

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

We know that "the ball is red or the ball is not red" is true because it has the form of the Law of Excluded Middle.

Example. Show that $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology. We see that the truth table for the propositional form is

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \vee Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
F	T	T	T	F	F	T
T	F	T	F	T	F	T
F	F	F	T	T	T	T

Thus $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

A **contradiction** is the negation of a tautology. Thus, $\sim(P \vee \sim P)$ is a contradiction. The negation of a contradiction is, of course, a tautology.

Conjunction, disjunction, and negation are very important in mathematics. Two other important connectives, the conditional and biconditional, will be studied in the next section. Other connectives having two components are not as useful in mathematics, but some are extremely important in digital computer circuit design.

Exercises 1.1

1. Which of the following are propositions?
   - (a) Where are my car keys?
   - (b) Christopher Columbus wore red boots at least once.
   - ★ (c) The national debt of Poland in 1938 was \$2,473,596.38.
   - (d) $x^2 \geqslant 0$.
   - ★ (e) Between January 1, 2205 and January 1, 2215, the population of the world will double.
   - (f) There are no zeros in the decimal expansion of π.
   - ★ (g) She works in New York City.
   - (h) There are more than 5 false statements in this book and this statement is one of them.
   - (i) There are more than 5 false statements in this book and this statement is not one of them.
2. Make truth tables for each of the following propositional forms.
   - ★ (a) $P \wedge \sim P$
   - (b) $P \vee \sim P$
   - ★ (c) $P \wedge (Q \vee R)$
   - (d) $(P \wedge Q) \vee (P \wedge R)$
   - ★ (e) $P \wedge \sim Q$
   - (f) $P \wedge (Q \vee \sim Q)$
   - ★ (g) $(P \wedge Q) \vee \sim Q$
   - (h) $\sim(P \wedge Q)$
   - (i) $(P \vee \sim Q) \wedge R$
   - (j) $\sim P \wedge \sim Q$
   - (k) $P \wedge P$
   - (l) $(P \wedge Q) \vee (R \wedge \sim S)$