An Introduction to

Mathematical Statistics

WITH A THOROUGH, MODERN TREATMENT OF PROBABILITY

Professor H. D. Brunk

AN INTRODUCTION

TO

Mathematical Statistics

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Preface

The purpose of this text is to assist the student to acquire as thorough an understanding of basic concepts in probability and statistics as is compatible with his mathematical background and the time available. The aim is not only to introduce him to mathematical statistics, but also to help prepare him for further study if he desires to undertake it. Thus, concepts which the author regards as fundamental are discussed thoroughly, while the student is given only a brief introduction to specialized methods.

This text was written originally for use in a one-semester, three-hour course introductory to mathematical statistics, offered to students who have studied the calculus but who have no formal background in probability or mathematical statistics. To make the book useful for more extended courses, however, additional material has been included. This additional material is presented in starred chapters and sections which are largely independent of one another; however, all presuppose the material presented in unstarred sections in the first 92 pages. It will be seen that a desirable feature of the book is its flexibility, which permits the teacher to shape the course according to the special needs of his students and his personal preferences.

The unstarred sections in the first 92 pages cover what appears to the author to be nearly the irreducible minimum background in probability for an understanding of the basic concepts of statistical inference; it has been his practice to devote the first six or seven weeks of the semester to this material. The remainder of the semester has usually been devoted to topics in the other unstarred chapters and sections. The teacher may wish, depending on his class and on the time available, to omit some of the proofs, particularly some of those in starred chapters and sections which make greater demands on the mathematical maturity of the student.

A suggested one-semester course includes Chapters 1 through 10, 12, or 13, omitting starred sections. Either or both of Chapters 16 or 17 or selected portions of them may be included in place of one or more of Chapters 10, 12, or 13.

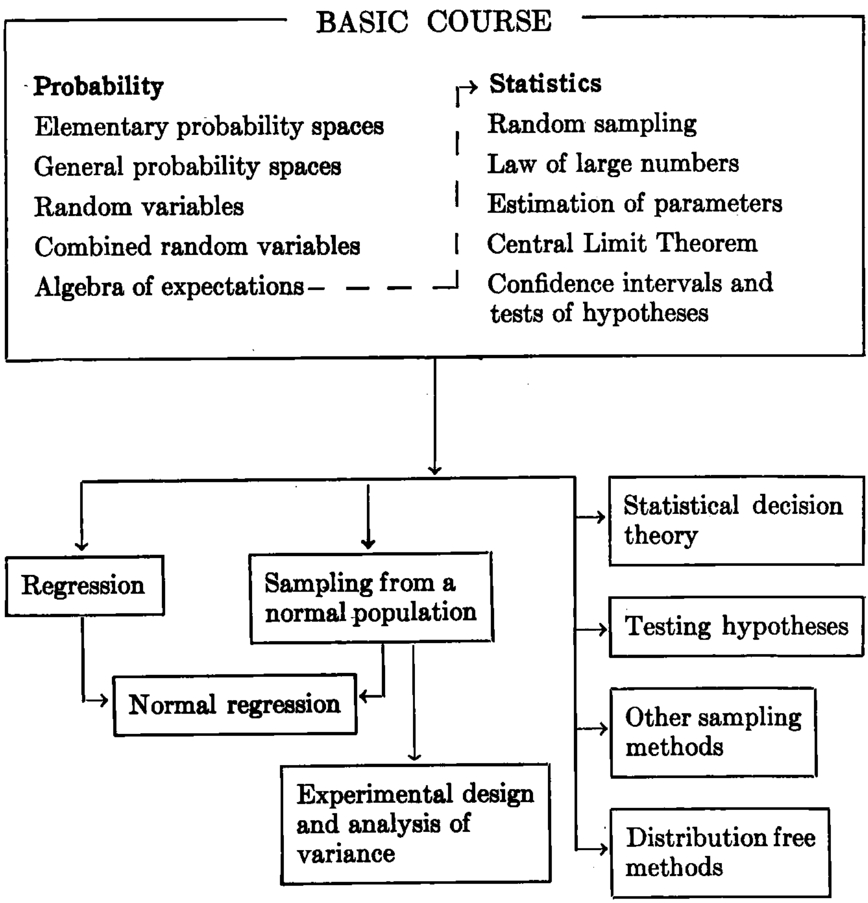
The teacher who wishes to take up some of the material in starred chapters or sections may do so in any order he wishes, except that the material on analysis of variance should be preceded by the chapter on sampling from a normal population, and the discussion of normal regression should be preceded by the chapter on regression as well as that on sampling from a normal population (cf. Organization Chart on page xi following Table of Contents). Also, the later chapters utilize the χ^2 distribution, which is defined in the chapter on sampling from a normal population. However, for use with those later chapters, the definition of the χ^2 distribution may be used without studying the chapter in which it appears.

Controversial questions in the foundations of probability are avoided; probability spaces are introduced as useful mathematical models. Inevitably, the teacher will feel that some topics have been omitted or slighted which he would like to include or stress. In particular, there is little work on multivariate analysis, a study of which the author feels should be preceded by more work in mathematics (in particular, matrix algebra) than is prerequisite for the course for which this text has been designed.

The author wishes to express his gratitude to the reviewers for the publisher, whose comments were very helpful; to Professor Frederick Mosteller in particular, for his many helpful suggestions for addition and revision which have markedly improved the book and made it more useful. Dr. Churchill Eisenhart, Dr. W. J. Youden, and their colleagues in the National Bureau of Standards were most helpful in making available compilations of published work of the Bureau for use as sources of problems. The author is grateful to Mr. James K. Yarnold, who prepared a number of the problems based on data taken from the literature. He is grateful also to his colleagues Professor G. B. Collier for the use of his problem lists, and Professor P. B. Burcham for his help and encouragement. The author is further indebted to Professor Sir Ronald A. Fisher, Cambridge, to Dr. Frank Yates, Rothamsted, and to Messrs. Oliver and Boyd Ltd., Edinburgh, for permission to reprint Table No. VII, from their book *Statistical Tables for Biological, Agricultural, and Medical Research*. The publishers also deserve the author's gratitude for their untiring efforts toward the goal of producing a text of maximal utility to teacher and student.

H. D. Brunk

ORGANIZATION CHART



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Part One

PROBABILITY

Introduction

Most of us manage to gain some experience with various games of chance in one way or another, and have some intuitive grasp of the kind of question with which probability theory has to deal. These concepts, made precise and integrated into a working theory—probability theory—have a great many applications in many fields, and in particular, in statistics. Indeed, probability forms the basis of the theory of statistical inference; it provides the conceptual framework within which the ideas of statistical inference may be discussed. Therefore the first part of this book is devoted to ideas arising in the theory of probability which are essential for an understanding of mathematical statistics.

Applications of a mathematical theory are made through mathematical models. That is, when faced with a "real," concrete, or physical situation to which he hopes to apply a mathematical theory, an investigator begins by idealizing the situation so as to make a mathematical model. For example, suppose a surveyor wishes to use measurements involving three trees, two on his side of a river and a third on the opposite bank, to determine the width of the river. He begins by constructing a mathematical model, in this case a geometric one. He idealizes the trees, replacing them by points, and introduces certain ideal elements, lines, which join them. He makes use of the geometric concepts of distance and angle, which have precise meanings only relative to his model and not relative to the actual existing situation. He makes readings on his instruments and incorporates the numerical results of his readings into his model as appropriate distances and angles. He then applies directly to his model the theory of plane geometry (or trigonometry).

The investigator (statistican or other) who wishes to apply the theory of probability must proceed in a similar fashion. The basic element in a mathematical model designed for the application of probability is called a probability space. The simplest of these we shall call elementary probability spaces. We shall state what we mean by an elementary probability space, and indicate some situations in which such a mathematical model seems to be (and has proved to be) appropriate. Having thus gained some experience with probabilistic concepts, we shall turn to more general probability spaces, and list axioms for the undefined elements (event, probability, etc.) of probability theory in the same spirit as one does for the undefined elements (point, line, etc.) of the theory of geometry. We shall continue throughout the book to discuss special situations in which probability models have proved useful, and shall not commit ourselves to any very general propositions concerning the utility or non-utility of such models.

CHAPTER
1

Elementary Probability Spaces

1. Introduction. When we wish to analyze mathematically a given situation, we have to begin by idealizing the situation, that is, by building a mathematical model. Consider, for example, the experiment of tossing a coin. Our particular interest is focused on the features that there are exactly two results possible, heads and tails, and that these are equally likely. These features are emphasized in the construction of a model, consisting of two *elementary events,* say H and T, and a probability $\frac{1}{2}$ associated with each. In general,

▶ *If an (idealized) proposed experiment can result in any of an exhaustive set of N equally likely and mutually exclusive possibilities, then an appropriate mathematical model is a set of N elementary events, a probability $1/N$ being associated with each.*

This ratio, $1/N$, is referred to as the probability that a performance of the experiment will result in a particular, specified one of the possibilities. We may also be interested in the occurrence of at least one of a specified subcollection of the possibilities. Such a subcollection is called an *event,* and its probability is defined to be the sum of the probabilities of the elementary events in the subcollection. The elementary events of this subcollection, we say, are *favorable* to the event, and we say the event *occurs* if the performance of the experiment results in the occurrence of one of the possibilities of that subcollection.

EXAMPLE 1. Suppose you hold a die, and propose to cast it. Thinking about the experiment before you perform it, you will set up a mathematical model containing 6 elementary events, a probability 1/6 being associated with each. The *event* consisting of the elementary events 2, 4, 6 might be described as "the event that an even number of spots will turn up." Three elementary events are favorable to this event, and its probability is accordingly 3/6, or 1/2. You might then say that you have a 50–50 chance of throwing an even number of spots.