LINEAR ALGEBRA

An Introduction

Paul J. Knopp

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an introduction

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to my wife
Margaret

Preface

This book presents the essential ideas of elementary linear algebra to students at the freshman-sophomore level. There are many students who need linear algebra, but who will not need calculus or differential equations; and even for those who do need both calculus and linear algebra, a case can be made that linear algebra should precede calculus or at least be taken simultaneously with the first semester of calculus. For these and other reasons, this book does not follow the practices of many other linear algebra texts.

First, no calculus experience is required for this book. Consequently, no function spaces are offered as examples of vector spaces, and no applications to differential equations are presented. The primary objects of study are the vector spaces $\mathbb{R}^n$, linear transformations between these spaces, real matrices, and, in the last chapter, quadratic functions. Discussions have been confined to these topics for several reasons. One is that freshmen and sophomores generally find other vector spaces bewildering. Another reason is that I am convinced that a teacher has done an outstanding teaching job if he gives students at this level a good understanding of linear functions and ample practice with the appropriate computational skills. A third reason is my belief that an understanding of linear functions between the spaces $\mathbb{R}^n$ can make many topics in calculus more accessible.

A second difference between this book and some others is its attitude toward mathematical proofs. Many feel that linear algebra should initiate students into the deductive style of higher level mathematics courses. This view has then influenced their style of presentation. My conviction is that the ideas of linear algebra are valuable for many students who will never progress to higher level mathematics courses or who will never find a need for the formal style of communication usually associated with courses in abstract mathematics. For this reason I have attempted to illustrate ideas and facts with numerical examples and then to discuss them in a more general setting, thus employing an inductive presentation rather than a deductive one. This, I believe, is appropriate for most students at this level. I have not labeled the reasons for a proposition as a proof even though in most cases they actually constitute a proof. On the other hand, I have not been content only with an appeal to numerical examples. The numerical examples are followed by a general formulation to help students make the transition from the particular to the general.

A third departure from the current practice is that set theoretic notation and language has been avoided wherever possible. This was done to make the text accessible to the average student reader. An instructor can easily introduce as much set theoretic notation as he wishes.

Even though the presentation is more inductive than is generally the case in linear algebra texts, still the book can be used for a rather standard course. The instructor will have to supply a few arguments which depend upon mathematical induction, and he will have to insist upon the level of formality he desires in the students' work. Nevertheless, it is my view that students will more quickly learn to state their ideas correctly if they are taught to check them with numerical examples; they must have experience with many examples before they can understand abstractions.

The text is written with the freshman in mind. It is not intended only for mathematics majors, nor is it intended only for nonmathematics majors; it is designed for any freshman student who needs the ideas and techniques of elementary linear algebra. Except for the last chapter, the text does not assume even analytic geometry beyond having some familiarity with obtaining equations for lines and planes. It does not even assume many of the familiar topics from high school algebra. It does assume, however, that the student can do arithmetic with real numbers. Consequently, as far as prerequisites are concerned, the text may fairly be designated a freshman text. The presentations are direct, and the Brief Exercises allow the student to test his understanding of the material immediately preceding them.

This is a mathematics text in that mathematical ideas are the backbone of the book; it is not concerned solely with techniques. Because of this, it is suitable for a first linear algebra course in colleges or in a transfer-oriented curriculum in two-year colleges.

For a one-semester course at the freshman-sophomore level, the first five chapters and the first three sections of Chapter 6 are probably sufficient. If the instructor wishes to include some material on the dot product from Chapter 8, then he may wish to omit the last two sections of Chapter 5 and the final section of Chapter 6. Chapters 7, 8, 9, and 10 are included so that a sophomore-junior level course can also be offered using the text.

Many people have helped me during the preparation of the manuscript. Dennison Brown used an early version in one of his classes, and Richard Sinkhorn gave me continued encouragement for the project. A number of reviewers were helpful. I want to thank the following gentlemen for valuable criticisms and suggestions: Meyer W. Belovicz, Joseph T. Buckley, Jack E. Forbes, Walter J. Gleason, David G. Mead, Richard C. Metzler, and Dieter Schmidt. Also thanks to Joseph D. Thibodeaux for helping me check the arithmetic in the manuscript. But especially I am indebted to my close friend Victor M. Manjarrez for using an early version in class and for many thoughtful conversations. He has greatly influenced my attitude toward mathematical education. The generosity of Mrs. Kathy Magann with her time in doing a fine emergency typing job was crucial in carrying the project to its conclusion. Also, a sincere word of gratitude to my friend Albert F. Thomasson for his encouragement throughout the years.

To my family I owe the greatest thanks. My wife, Margaret, cheerfully encouraged me during the writing, and my children, Cynthia, Elizabeth, and John, were especially understanding in "playing somewhere else." Without their help and support this book could never have been completed.

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Houston, Texas *Paul J. Knopp*

About the Author

Paul J. Knopp was born in San Antonio, Texas, on Janurary 3, 1934. He received his B.S. at Spring Hill College in 1957, his A.M. at Harvard University in 1958, and his Ph.D. at the University of Texas in 1962. He has taught at Spring Hill College (1959), the University of Missouri (1962–1964), and the University of Houston (1964–). In 1970 he was a recipient of the Teaching Excellence Award at the University of Houston. In 1971–1973 he served as Associate Director and later as Executive Director of the Committee on the Undergraduate Program in Mathematics (CUPM). He is a member of the Mathematical Association of America and of the American Mathematical Society. He has published research articles dealing with problems in matrix theory and analysis.

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Introduction to Systems of Linear Equations

SECTION 1-1
Systems of Equations and the Use of Matrices to Solve Them

PREVIEW *In this section we discuss methods for finding solutions to systems of linear equations. The procedure involves three different operations: (a) interchange of two equations, (b) multiplication of an equation by a nonzero real number, and (c) addition of a multiple of one equation to another. Finally, we show how our computations can be handled in a more convenient way by using matrices.*

First we study systems of linear equations. An example is

$$\begin{aligned} x - y &= 2 \\ 3x + 2y &= 5. \end{aligned}$$

The letters x and y represent unknown numbers (unknowns). The numbers 2 and 5 on the right-hand side of the equations are called constants. The numbers multiplying x and y are called the coefficients of x and y. To determine the coefficients, write the equations with a plus sign between the terms. Thus the coefficients of x are 1 in the first equation and 3 in the second. Since $x - y = x + (-1)y$, -1 is the coefficient of y in the first equation, and 2 is the coefficient of y in the second.

We have written the unknowns on the left-hand side of the equations and the constants on the right-hand side, but this is only a convention. We could just as well have written $x = 2 + y$ or $-y = 2 - x$ or $x - y - 2 = 0$. These are different forms of the same equation. However, we will usually write the unknowns on the left of the equality and the constants on the right.

Let us consider another example.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + x_3 &= 5 \end{aligned}$$

This is a system of two equations in the three unknowns x_1, x_2, and x_3. The coefficients of x_1 are 2 and 1, the coefficients of x_2 are 3 and 0, and the coefficients of x_3 are -1 and $+1$ in the first and second equations, respectively. The constants are 1 and 5. In this example we distinguish between different unknowns with subscripts on the same letter x, whereas in the first example we used different letters for different unknowns. Both procedures are common; ordinarily we use different letters if there are only a few unknowns and subscripts if there are many.

Let us consider a third example.

$$\begin{aligned} 5x + 2y - z &= 0 \\ x + 3y + 4z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

This is a system of three equations in three unknowns. This system is called *homogeneous* because the constants are all zero; if at least one constant is different from zero, then the system is called *nonhomogeneous*.

The goal of this section is to give procedures for finding all the solutions for a system of equations. A solution for

$$\begin{aligned} x - y &= 2 \\ 3x + 2y &= 5 \end{aligned} \tag{1}$$

consists of two numbers, one for x and one for y, which satisfy all the equations. Here, for example, $x = \frac{9}{5}$ and $y = -\frac{1}{5}$ constitute a solution. (In fact, this is the only solution.) A solution for

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + x_3 &= 5 \end{aligned} \tag{2}$$

consists of three numbers, one for each unknown. For example, $x_1 = 1$, $x_2 = 1$, $x_3 = 4$ is a solution. Another is $x_1 = 0$, $x_2 = 2$, $x_3 = 5$.

Rather than writing $x = \frac{9}{5}$ and $y = -\frac{1}{5}$, we abbreviate by saying that $(\frac{9}{5}, -\frac{1}{5})$ is a solution for the system (1). The first number is used for the first unknown and the second number for the second unknown. Using this kind of abbreviation, we say that (1,1,4) and (0,2,5) are solutions for system (2).

We call $(\frac{9}{5}, -\frac{1}{5})$ an *ordered* pair because the order is important: the order indicates which number corresponds to which unknown. Similarly (1,1,4) is an ordered triple. For a system having four unknowns, a solution is an ordered quadruple; for five unknowns we need an ordered quintuple; for more unknowns we call solutions 6-tuples, 7-tuples, etc.

Now we explain how to find all the solutions for any system of linear equations. Some systems have only one solution, others have many, whereas still others have none. We begin with an example.

Example Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x - 2y &= 4 \\ 3x + 2y &= 7. \end{aligned}$$

We describe a procedure for obtaining all the solutions of this system.

Eliminate x from the second and third equations by adding -2 times the first equation to the second and by adding -3 times the first equation to the third.

$$\begin{aligned} x - y &= 2 \\ 0x + 0y &= 0 \\ 5y &= 1 \end{aligned}$$

Interchange the position of the second and third equations.

$$\begin{aligned} x - y &= 2 \\ 5y &= 1 \\ 0x + 0y &= 0 \end{aligned}$$

Multiply the third equation by $\frac{1}{5}$.

$$\begin{aligned} x - y &= 2 \\ y &= \tfrac{1}{5} \\ 0x + 0y &= 0 \end{aligned}$$

Eliminate y from the first equation by adding the second equation to the first.

$$\begin{aligned} x &= \tfrac{11}{5} \\ y &= \tfrac{1}{5} \\ 0x + 0y &= 0 \end{aligned}$$

In the example above we go from one system to another by using three operations:

(1) interchange of two equations,
(2) multiplication of an equation by a nonzero real number, and
(3) addition of a multiple of one equation to another.

Using these operations we can find the solutions for (solve) any system of linear equations. We describe a **systematic procedure** which always works. (In this description we label the unknowns x_1, x_2, and so on.)

Step 1: Isolate $1 \cdot x_1$ in the first equation as follows:
- (a) locate an equation in which x_1 appears,
- (b) place this equation first,
- (c) obtain $1 \cdot x_1$ in the first equation by multiplying the first equation by the appropriate number, and
- (d) eliminate x_1 from all the other equations by adding multiples of the first equation to the other equations.