Introduction to

PROBABILITY MODELS

Fifth Edition

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Preface to the Fifth Edition

The fifth edition includes additional material in all chapters, with the greatest number of additions in Chapters 5 and 10. For instance, new examples relating to analyzing greedy algorithms, minimizing highway encounters, collecting coupons, and tracking the AIDS virus, as well as additional material on compound Poisson processes appear in Chapter 5. Chapter 10 includes new material on the theory of options pricing. The arbitrage theorem is presented and its relationship to the duality theorem of linear program is indicated. We show how the arbitrage theorem leads to the Black–Scholes option pricing formula.

This edition also contains over 120 new exercises. There are two solutions manuals for the text. One manual, which contains the solutions of all the exercises of the text, is available only to instructors. In addition, over 100 exercises in the text are starred, and their solutions are available to students in a separate solutions manual.

Preface to the Fourth Edition

This text is intended as an introduction to elementary probability theory and stochastic processes. It is particularly well-suited for those wanting to see how probability theory can be applied to the study of phenomena in fields such as engineering, management science, the physical and social sciences, and operations research.

It is generally felt that there are two approaches to the study of probability theory. One approach is heuristic and nonrigorous and attempts to develop in the student an intuitive feel for the subject which enables him or her to "think probabilistically." The other approach attempts a rigorous development of probability by using the tools of measure theory. It is the first approach that is employed in this text. However, as it is extremely important in both understanding and applying probability theory to be able to "think probabilistically," this text should also be useful to students interested primarily in the second approach.

Chapters 1 and 2 deal with basic ideas of probability theory. In Chapter 1 an axiomatic framework is presented, while in Chapter 2 the important concept of a random variable is introduced.

Chapter 3 is concerned with the subject matter of conditional probability and conditional expectation. "Conditioning" is one of the key tools of probability theory, and it is stressed throughout the book. When properly used, conditioning often enables us to easily solve problems that at first glance seem quite difficult. The final section of this chapter, not in the original edition, presents applications to (1) a computer list problem, (2) a random graph, and (3) the Polya urn model and its relation to the Bose–Einstein distribution.

In Chapter 4 we come into contact with our first random, or stochastic, process, known as a Markov chain, which is widely applicable to the study of many real-world phenomena. New applications to generics and production processes are presented. The concept of time reversibility is introduced and its usefulness illustrated. In the final section we consider a model for optimally making decisions known as a Markovian decision process.

In Chapter 5 we are concerned with a type of stochastic process known as a counting process. In particular, we study a kind of counting process known as a Poisson process. The intimate relationship between this process and the exponential distribution is discussed.

Chapter 6 considers Markov chains in continuous time with an emphasis on birth and death models. Time reversibility is shown to be a useful concept, as it is in the study of discrete-time Markov chains. The final section presents the computationally important technique of uniformization.

Chapter 7, the renewal theory chapter, is concerned with a type of counting process more general than the Poisson. By making use of renewal reward processes, limiting results are obtained and applied to various fields.

Chapter 8 deals with queueing, or waiting line, theory. After some preliminaries dealing with basic cost identities and types of limiting probabilities, we consider exponential queueing models and show how such models can be analyzed. Included in the models we study is the important class known as network of queues. We then study models in which some of the distributions are allowed to be arbitrary.

Chapter 9 is concerned with reliability theory. This chapter will probably be of greatest interest to the engineer and operations researcher.

Chapter 10 deals with Brownian motion and Chapter 11 with simulation.

Ideally, this text would be used in a one-year course in probability models. Other possible courses would be a one-semester course in introductory probability theory (involving Chapters 1–3 and parts of others) or a course in elementary stochastic processes. It is felt that the textbook is flexible enough to be used in a variety of possible courses. For example, I have used Chapters 5 and 8, with smatterings from Chapters 4 and 6, as the basis of an introductory course in queueing theory.

There are many examples worked out throughout the text, and there are also a large number of problems to be worked by students. Answers to selected problems appear in the text, and a separate solutions manual is available to instructors using the text.

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3. Conditional Probability and Conditional Expectation

4. Markov Chains

5. The Exponential Distribution and the Poisson Process

6. Continuous-Time Markov Chains

7. Renewal Theory and Its Applications

8. Queueing Theory

9. Reliability Theory

10. Brownian Motion and Stationary Processes

11. Simulation

Chapter 1
Introduction to Probability Theory

1.1. Introduction

Any realistic model of a real-world phenomenon must take into account the possibility of randomness. That is, more often than not, the quantities we are interested in will not be predictable in advance but, rather, will exhibit an inherent variation that should be taken into account by the model. This is usually accomplished by allowing the model to be probabilistic in nature. Such a model is, naturally enough, referred to as a probability model.

The majority of the chapters of this book will be concerned with different probability models of natural phenomena. Clearly, in order to master both the "model building" and the subsequent analysis of these models, we must have a certain knowledge of basic probability theory. The remainder of this chapter, as well as the next two chapters, will be concerned with a study of this subject.

1.2. Sample Space and Events

Suppose that we are about to perform an experiment whose outcome is not predictable in advance. However, while the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by S.

Some examples are the following.

1. If the experiment consists of the flipping of a coin, then

$$S = \{H, T\}$$

where H means that the outcome of the toss is a head and T that it is a tail.

2. If the experiment consists of tossing a die, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

where the outcome i means that i appeared on the die, $i = 1, 2, 3, 4, 5, 6$.

3. If the experiment consists of flipping two coins then the sample space consists of the following four points

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The outcome will be (H, H) if both coins come up heads; it will be (H, T) if the first coin comes up heads and the second comes up tails; it will be (T, H) if the first comes up tails and the second heads; and it will be (T, T) if both coins come up tails.

4. If the experiment consists of tossing two dice, then the sample space consists of the 36 points

$$S = \left\{\begin{matrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{matrix}\right\}$$

where the outcome (i, j) is said to occur if i appears on the first die and j on the second die.

5. If the experiment consists of measuring the lifetime of a car, then the sample space consists of all nonnegative real numbers. That is,

$$S = [0, \infty)^* \quad ◆$$

Any subset E of the sample space S is known as an *event*. Some examples of events are the following.

1′. In Example (1), if $E = \{H\}$, then E is the event that a head appears on the flip of the coin. Similarly, if $E = \{T\}$, then E would be the event that a tail appears.

* The set (a, b) is defined to consist of all points x such that $a < x < b$. The set $[a, b]$ is defined to consist of all points x such that $a \leq x \leq b$. The sets $(a, b]$ and $[a, b)$ are defined, respectively, to consist of all points x such that $a < x \leq b$ and all points x such that $a \leq x < b$.

2′. In Example (2), if $E = \{1\}$, then E is the event that one appears on the toss of the die. If $E = \{2, 4, 6\}$, then E would be the event that an even number appears on the toss.

3′. In Example (3), if $E = \{(H, H), (H, T)\}$, then E is the event that a head appears on the first coin.

4′. In Example (4), if $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, then E is the event that the sum of the dice equals seven.

5′. In Example (5), if $E = (2, 6)$, then E is the event that the car lasts between two and six years. ◆

For any two events E and F of a sample space S we define the new event $E \cup F$ to consist of all points which are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if *either* E or F occurs. For example, in (1) if $E = \{H\}$ and $F = \{T\}$, then

$$E \cup F = \{H, T\}$$

That is, $E \cup F$ would be the whole sample space S. In (2) if $E = \{1, 3, 5\}$ and $F = \{1, 2, 3\}$, then

$$E \cup F = \{1, 2, 3, 5\}$$

and thus $E \cup F$ would occur if the outcome of the die is either a 1 or 2 or 3 or 5. The event $E \cup F$ is often referred to as the *union* of the event E and the event F.

For any two events E and F, we may also define the new event EF, referred to as the *intersection* of E and F, as follows. EF consists of all points which are *both* in E and in F. That is, the event EF will occur only if E and F occur. For example, in (2) if $E = \{1, 3, 5\}$ and $F = \{1, 2, 3\}$, then

$$EF = \{1, 3\}$$

and thus EF would occur if the outcome of the die is either 1 or 3. In example (1) if $E = \{H\}$ and $F = \{T\}$, then the event EF would not consist of any points and hence could not occur. To give such an event a name we shall refer to it as the null event and denote it by $\varnothing$. (That is, $\varnothing$ refers to the event consisting of no points.) If $EF = \varnothing$, then E and F are said to be *mutually exclusive*.

We also define unions and intersections of more than two events in a similar manner. If $E_1, E_2, \ldots$ are events, then the union of these events, denoted by $\bigcup_{n=1}^{\infty} E_n$, is defined to be that event which consists of all points that are in E_n for at least one value of $n = 1, 2, \ldots$. Similarly, the intersection of the events E_n, denoted by $\prod_{n=1}^{\infty} E_n$, is defined to be the event consisting of those points that are in all of the events E_n, $n = 1, 2, \ldots$.

Finally, for any event E we define the new event E^c, referred to as the *complement* of E, to consist of all points in the sample space S which are not in E. That is E^c will occur if and only if E does not occur. In Example (4) if $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, then E^c will occur if the sum of the dice does not equal seven. Also note that since the experiment must result in some outcome, it follows that $S^c = \varnothing$.

1.3. Probabilities Defined on Events

Consider an experiment whose sample space is S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three conditions:

(i) $0 \leq P(E) \leq 1$.
(ii) $P(S) = 1$.
(iii) For any sequence of events $E_1, E_2, \ldots$ which are mutually exclusive, that is, events for which $E_n E_m = \varnothing$ when $n \neq m$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

We refer to $P(E)$ as the probability of the event E.

Example 1.1 In the coin tossing example, if we assume that a head is equally likely to appear as a tail, then we would have

$$P(\{H\}) = P(\{T\}) = \tfrac{1}{2}$$

On the other hand, if we had a biased coin and felt that a head was twice as likely to appear as a tail, then we would have

$$P(\{H\}) = \tfrac{2}{3}, \qquad P(\{T\}) = \tfrac{1}{3} \quad \blacklozenge$$

Example 1.2 In the die tossing example, if we supposed that all six numbers were equally likely to appear, then we would have

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \tfrac{1}{6}.$$

From (iii) it would follow that the probability of getting an even number would equal

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\})$$
$$= \tfrac{1}{2} \quad \blacklozenge$$

Remark We have chosen to give a rather formal definition of probabilities as being functions defined on the events of a sample space. However, it turns out that these probabilities have a nice intuitive property. Namely, if our experiment is repeated over and over again then (with probability 1) the proportion of time that event E occurs will just be $P(E)$.

Since the events E and E^c are always mutually exclusive and since $E \cup E^c = S$ we have by (ii) and (iii) that

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

or

$$P(E) + P(E^c) = 1 \tag{1.1}$$

In words, Equation (1.1) states that the probability that an event does not occur is one minus the probability that it does occur.

We shall now derive a formula for $P(E \cup F)$, the probability of all points either in E or in F. To do so, consider $P(E) + P(F)$, which is the probability of all points in E plus the probability of all points in F. Since any point that is in both E and F will be counted twice in $P(E) + P(F)$ and only once in $P(E \cup F)$, we must have

$$P(E) + P(F) = P(E \cup F) + P(EF)$$

or equivalently

$$P(E \cup F) = P(E) + P(F) - P(EF) \tag{1.2}$$

Note that in the case that E and F are mutually exclusive (that is, when $EF = \varnothing$), then Equation (1.2) states that

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(\varnothing) \\ &= P(E) + P(F) \end{aligned}$$

a result which also follows from condition (iii). (Why is $P(\varnothing) = 0$?)

Example 1.3 Suppose that we toss two coins, and suppose that we assume that each of the four points in the sample space

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

are equally likely and hence have probability $\frac{1}{4}$. Let

$$E = \{(H, H), (H, T)\} \qquad \text{and} \qquad F = \{(H, H), (T, H)\}$$

That is, E is the event that the first coin falls heads, and F is the event that the second coin falls heads.