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MATHEMATICS LECTURE NOTE SERIES

Algebraic Topology

A First Course

Marvin J. Greenberg
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ALGEBRAIC TOPOLOGY

A First Course

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1981

THE BENJAMIN/CUMMINGS PUBLISHING COMPANY
Advanced Book Program/World Science Division
Reading, Massachusetts

London · Amsterdam · Don Mills, Ontario · Sydney · Tokyo

Library of Congress Cataloging in Publication Data

Greenberg, Marvin J.
Algebraic topology.

(Mathematics lecture note series ; 58)
"A revision of the first author's Lectures on algebraic topology"—P.
Bibliography: p.
Includes index.
1. Algebraic topology. I. Harper, John R., 1941- . II. Title. III. Series.
QA612.G7 514'.2 81-17108
ISBN 0-8053-3558-7 AACR2
ISBN 0-8053-3557-9 (pbk.)

American Mathematical Society (MOS) Subject Classification Scheme (1980): 55-01, 57-01

Manufactured in the United States of America

ABCDEFGHIJ-HA-8987654321

PREFACE

Algebraic Topology is one of the major creations of twentieth-century mathematics. Its influence on other parts of mathematics, such as algebra [38], number theory [4, 49], algebraic geometry [27, 31, 50], differential geometry [26], and analysis [12, 1963–64] has been enormous. In its own right, it is a major tool for the investigation of topological spaces, especially manifolds. Its key idea is to attach algebraic structures to topological spaces and their maps in such a way that the algebra is both invariant under a variety of deformations of spaces and maps, and computable.

This book is intended as a first course, sufficiently comprehensive to enable the student either to use the subject in other fields of endeavor and/or to pursue its development and applications in more advanced texts and the literature.

Our presentation is a revision of the first author's *Lectures on Algebraic Topology*. The intent in revising was to make those additions of theory, examples, and exercises which updated, enhanced, and simplified the original exposition. The point of view and organizational principles of the earlier book have been maintained. Virtually all of the original book has been reproduced.

In the additional material, special attention has been given to calculations, with more geometry to balance all the algebra.

There are essentially four parts to this work: Sections 1–7 form Part I, elementary homotopy theory. Homotopy of paths and maps is defined, and the fundamental group is constructed. The classification of covering spaces by means of subgroups of the fundamental group is given, and, finally, the higher homotopy groups are defined inductively using loop spaces, following Hurewicz.

Sections 8–21, Part II, treat singular homology theory. This Part has been influenced by the lucid notes of E. Artin [3] and the work of Eilenberg-Steenrod [23]. The advantages of singular over simplicial homology theory are that, first, it applies to arbitrary topological spaces; second, it is obviously topologically invariant; third, once the excision theorem is proved, there is never again any need to subdivide, and, finally, it is easier to calculate once the basic formulas (19.16–19.18) have been proved. Combinatorial techniques are still very important in algebraic topology [36, 62, 70]. However, it is now recognized that algebraic topology encompasses at least three different categories—topological, differential, and piecewise linear. In this book we treat primarily the first (references for the second are [15, 17, 41–44, 51, 55, 68, 71]). The classical applications of homology theory to spheres are given in Sections 15, 16, and 18.

Sections 22–28 form Part III, the orientability and duality properties of manifolds. This part has been greatly influenced by notes of Dold, Puppe, and Milnor. No assumption of triangulability is needed in this treatment. The correct cohomology theory for the duality is that of Alexander-Cech; however, for brevity's sake, we only describe the Alexander-Cech cohomology module of a subspace A as the inductive limit over the neighborhoods of A of the singular cohomology modules. We show that this coincides with the singular cohomology module when A is a compact ANR.

Finally in Part IV we develop the basic features of the theory of products in cohomology. The applications include the Lefschetz fixed point theorem for compact oriented manifolds and an introduction to intersection theory in closed manifolds.

Each part is divided into several sections. These are the organizational units of the text. There is considerable flexibility (especially in the latter parts) in the order in which they may be studied. In Part II, many sections conclude with material which may be skimmed or skipped at first reading.

Most sections end with sets of exercises. No theoretical development depends on an exercise nor is further theoretical material given as exercises. Most exercises concern calculation and, as the subject develops, geometric applications are made. There are many cross-references among exercises. Refinements of calculations available with developments of the theory are offered. Similarly, improvements in geometric results are made in several sections. This process imitates the way the subject actually developed, and may help motivate the successive layers of abstraction through which the subject passes. Some exercises are accompanied with suggestions for their solution. These suggestions should not be taken too seriously. Most problems can be solved in different ways, and one's favorite solution may not receive widespread approval. But it is discouraging to be totally "stuck" so suggestions are offered to alleviate that condition.

Prerequisites for this book, besides the usual "mathematical maturity," are very few. In algebra, familiarity with groups, rings, modules, and their homomorphisms is required. From Section 20 on, some basic results for modules over principal ideal domains will be used. Only in Sections 29 and 30 is knowledge of the basic properties of the tensor product of two modules needed. The language of categories and functors is used throughout the book, although no theorems about categories are required. For all of this material, see Lang [35].

In point-set topology, the reader is presumed to be familiar with the basic facts about continuity, compactness, connectedness and pathwise-connectedness, product spaces, and quotient spaces. Only in the appendix to Section 26* do we require a nontrivial result, Tietze's extension theorem. Section 7 uses some elementary results about the compact-open topology on function spaces. For this material, see Dugundji [20] or Kelley [34].

I recommend the survey articles [44a, 62, and 75, pp. 227–31 and its bibliography] to the reader seeking further information on the extraordinary achievements in algebraic topology in recent years.

I thank M. Artin, H. Edwards, S. Lang, B. Mazur, V. Poenaru, H. Rosenberg, E. Spanier, and A. Vasquez; also my students Berkovits, Perry, and Webber, for helpful comments.

We are grateful to a number of people for helpful remarks concerning the revision. The comments of D. Anderson, E. Bishop, G. Carlsson, M. Friedman, T. Frankel, J. Lin, and K. Millett were helpful in deciding what to include and what to leave out. As the work developed, valuable remarks were made by M. Cohen, A. Liulevicius, R. Livesay, S. Lubkin, H. Miller, R. Mandelbaum, N. Stein, and A. Zabrodsky.

The typing of the manuscript was expertly done by S. Agostinelli, R. Colon, and M. Lind. Additional figures were drawn by D. McCumber.

Special thanks are extended to Doris, Jennifer, and Allison for not overreacting to neglect endured during preparation and assembly of this material.

Lastly, we thank Errett Bishop for suggesting that we collaborate on this book.

MARVIN J. GREENBERG
JOHN R. HARPER

*(26.17)

CONTENTS

Part 1
ELEMENTARY HOMOTOPY THEORY

Introduction to Part I

The wellspring of ideas leading to algebraic topology was the perception, developed largely in the latter half of the nineteenth century, that many properties of functions were invariant under "deformations." For example, Cauchy's theorem and the calculus of residues in complex analysis assert invariance of complex integrals with respect to continuous deformations of curves. Perhaps the true starting point was Riemann's theory of abelian integrals. It was here that the significance of the connectivity of surfaces was recognized. The interested reader is strongly encouraged to examine Felix Klein's exposition of Riemann's theory [80], during the study of algebraic topology.

It was Poincaré who first systematically attacked the problem of attaching numerical topological invariants to spaces. In his investigations, he perceived the difference between curves *deformable* to one another and curves *bounding* a larger space. The former idea led to the introduction of homotopy and the fundamental group, while the latter led to homology.

The development of these ideas into a mathematical theory is elaborate. However, the idea guiding the development is easily described. Certain functors are constructed. Thus to each topological space X is assigned a group $F(X)$, and to each map $f: X \to Y$ (a "map" of topological spaces will always mean a "continuous map" unless otherwise stated) is assigned a homomorphism $F(f): F(X) \to F(Y)$ such that

(1) If $Y = X$ and $f =$ identity, then $F(f) =$ identity,

(2) If $g: Y \to Z$, then $F(gf) = F(g)F(f)$.

Illustration: Suppose we have a diagram of topological spaces and maps

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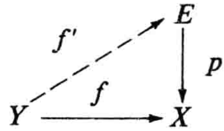
ISBN 0-8053-3557-9(Pbk)



and the problem is to find f' such that $pf' = f$. Applying the functor F we see that a necessary condition for a solution to exist is that $F(f)$ send $F(Y)$ into the subgroup $F(p)(F(E))$ of $F(X)$. In certain cases later we will see this is also sufficient (6.1).

Illustration: Suppose $f : X \to Y$ is a homeomorphism. Then by functoriality $F(f^{-1})$ is inverse to $F(f)$, so that $F(f)$ is an isomorphism. Thus a *necessary* condition (but usually not sufficient) that X and Y be homeomorphic is that $F(X)$ and $F(Y)$ be isomorphic groups. This is usually the easiest way to prove that two given spaces with similar topological properties are not homeomorphic.

Illustration: Suppose $i : A \to X$ is the inclusion map of a subspace A into X and our problem is to find a map $r : X \to A$ such that ri is the identity map of A (such a map r is called a *retraction* of X onto A). By functoriality, $F(r)F(i)$ equals the identity transformation of $F(A)$, so that $F(i)$ sends $F(A)$ isomorphically onto a subgroup of $F(X)$. If we happen to know, e.g., that $F(X)$ is trivial while $F(A)$ is not, it then follows that no retraction can exist. This is the way the Brouwer Fixed Point Theorem is proved (4.11 and 15.7).

The reader may construct some more illustrations to convince himself of the fruitfulness of this point of view.

1. Arrangement of Part I

In Part I, we treat the fundamental group and the closely related notion of covering space. The geometric idea for the construction of the fundamental group functor is homotopy of paths. Roughly speaking, a homotopy of a path is a deformation leaving the end points fixed. A composition of paths may be defined when the end point of one agrees with the initial point of the other. Familiar algebraic properties, like associativity, do not hold, but do hold up to homotopy. The result is a group structure on equivalences classes, called the fundamental group. This group is not just a topological invariant, but invariant under a larger class of maps, called *homotopy equivalences*. These topics are treated in Sections 2 and 3.

In order to exploit the fundamental group, we must be able to calculate it. There are two principal routes to calculation: the Seifert-Van Kampen theorem and the use of covering spaces. The versions of the former used in this text are stated in (4.12). There are several excellent accounts available in other texts, so we do not reproduce the details. Our treatment of the fundamental group of the circle is the prototype for the theory of covering spaces. The lifting theorem for covering spaces (6.1), besides being useful, is an outstanding example of the blend of algebra and geometry that gives this subject its special flavor. Part I concludes with a brief discussion of higher homotopy groups, introduced by means of loop spaces.

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2. Homotopy of Paths

Consider, in the plane, the problem of integrating a function f of a complex variable around a closed curve C, e.g., the unit circle. We have, for example,

$$\int_C z\, dz = 1$$

$$\int_C \frac{dz}{z} \neq 0$$

What is the difference? We take the point of view that C can be "shrunk to a point" within the domain of analyticity of z (i.e., the whole plane), hence integrating around C is equivalent to integrating at a point, which gives 0. On the contrary C cannot be "shrunk to a point" within the domain of $1/z$.

More precisely, let σ, τ be *paths* in a space X (i.e., maps of the unit interval I into X) with the same end points (i.e., $\sigma(0) = \tau(0) = x_0$, $\sigma(1) = \tau(1) = x_1$). We say *$\sigma$ and τ are homotopic with end points held fixed* written

$$\sigma \simeq \tau \text{ rel } (0, 1)$$

if there is a map $F : I \times I \to X$ such that

(1) $F(s, 0) = \sigma(s)$ all s

(2) $F(s, 1) = \tau(s)$ all s

(3) $F(0, t) = x_0$ all t

(4) $F(1, t) = x_1$ all t

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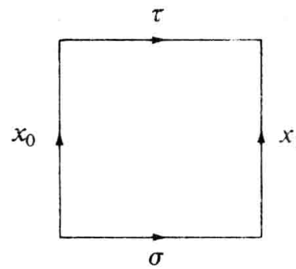
F is called a *homotopy* from σ to τ. For each t, $s \to F(s,t)$ is a path F_t from x_0 to x_1, and $F_0 = \sigma$, $F_1 = \tau$. We write

$$F_t : \sigma \simeq \tau \qquad \text{rel } (0, 1)$$

Pictorially:



In particular if σ is a *loop* at x_0 (i.e., $x_1 = x_0$) and τ is the constant loop $\tau(s) = x_0$ for all s, and if $\sigma \simeq \tau$ rel $(0, 1)$, we say that "σ can be shrunk to a point," or is *homotopically trivial*.

Then the correct statement of Cauchy's Theorem is that $\int_C f(z)dz = 0$ for all loops C in the domain X of analyticity of f which are homotopically trivial (more generally, homologically trivial).

The following properties of relation $\simeq$ are easily proved:

(1) $\sigma \simeq \sigma \quad \text{rel } (0, 1)$

(2) $\sigma \simeq \tau \quad \text{rel } (0, 1) \Rightarrow \tau \simeq \sigma \text{ rel } (0, 1)$

(3) $\sigma \simeq \tau \quad \text{rel } (0, 1) \text{ and } \tau \simeq \rho \text{ rel } (0, 1) \Rightarrow \sigma \simeq \rho \text{ rel } (0, 1)$

Thus we can consider the homotopy classes $[\sigma]$ of paths σ from x_0 to x_1 under the equivalence relation $\simeq$.

If σ is a path from x_0 to x_1 and τ is now taken to be a path from x_1 to x_2, we define a path $\sigma\tau$ from x_0 to x_2 by first travelling along σ, then along τ; more precisely we set

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & 0 \le t \le 1/2 \\ \tau(2t-1) & 1/2 \le t \le 1 \end{cases}$$

(4) $\sigma \simeq \sigma'$ rel $(0, 1)$ and $\tau \simeq \tau'$ rel $(0, 1) \Rightarrow \sigma\tau \simeq \sigma'\tau'$ rel $(0, 1)$.

Proof: If $F_t : \sigma \simeq \sigma'$ rel $(0, 1)$, $G_t : \tau \simeq \tau'$ rel $(0, 1)$, then

$$F_tG_t : \sigma\tau \simeq \sigma'\tau' \text{ rel } (0, 1).$$ ∎

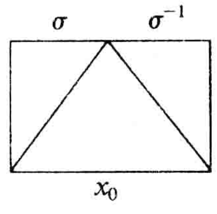
Thus we can multiply the *class* of σ on the right by the *class* of τ without ambiguity, always supposing the end point of σ equals the initial point of τ.

(2.1) *Theorem. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X at x_0. If multiplication in $\pi_1(X, x_0)$ is defined as above, $\pi_1(X, x_0)$ becomes a group, in which the neutral element is the class of the constant loop at x_0 and the inverse of a class $[\sigma]$ is the class of the loop σ^{-1} defined by*

$$\sigma^{-1}(t) = \sigma(1 - t) \qquad 0 \leq t \leq 1$$

(*i.e., travel backwards along σ*).

Proof. We will prove that $\sigma\sigma^{-1} \simeq x_0$, where now x_0 denotes also the constant loop at the point x_0. The homotopy is given by the following diagram:



Thus, we define $F(s, t)$ by

$$F(s, t) = \begin{cases} \sigma(2s) & 0 \leq 2s \leq t \\ \sigma(t) & t \leq 2s \leq 2 - t \\ \sigma^{-1}(2s - 1) & 2 - t \leq 2s \leq 2 \end{cases}$$

Clearly these functions are continuous on each triangle and they agree on the intersections, hence by an elementary argument F is continuous on the whole square.

The proof that multiplication is associative (up to homotopy) can be done similarly, as can the proof that the class of x_0 is the neutral element.

$$\text{Define } F(s,t) = \begin{cases} \sigma\left(\dfrac{4s}{t+1}\right) & 0 \le s \le \frac{1}{4}(t+1) \\ \tau(4s-t-1) & \frac{1}{4}(t+1) \le s \le \frac{1}{4}(t+2) \\ \omega\left(\dfrac{4s-t-2}{2-t}\right) & \frac{1}{4}(t+2) \le s \le 1 \end{cases}$$

to establish $(\sigma\tau)\omega \simeq \sigma(\tau\omega)$ rel $(0, 1)$.

$$\text{Define } F(s,t) = \begin{cases} \sigma\left(\dfrac{2s}{t+1}\right) & 0 \le s \le \dfrac{t+1}{2} \\ x_0 & \dfrac{t+1}{2} \le s \le 1 \end{cases}$$

to establish that the constant path at x_0 is the neutral element of the fundamental group. ■

Is there a relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$? There certainly is not if x_0 and x_1 lie in different path-connected components of X. However, we have the following result.

(2.2) *Proposition*. Let α *be a path from* x_0 *to* x_1. *The mapping* $[\sigma] \to [\alpha^{-1}\sigma\alpha]$ *is an isomorphism* α_* *of the group* $\pi_1(X, x_0)$ *onto* $\pi_1(X, x_1)$.

Proof. It is clearly a homomorphism, and $(\alpha^{-1})_*$ is its inverse (where α^{-1} is the path defined as in 2.1). ■

(2.3) *Corollary. If* X *is pathwise connected, the group* $\pi_1(X, x_0)$ *is independent of the point* x_0, *up to isomorphism.*

In that case we often write simply $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it *the fundamental group* of X.

We would like π_1 to be a functor from spaces to groups, but since $\pi_1(X, x_0)$ does depend on the base point x_0 in the general case, we must put the base points into our category if we are to obtain a functor. So define the category of *pointed topological spaces* to have as objects pairs (X, x_0), and as morphisms the maps $f: X \to Y$ such that $f(x_0) = y_0$. For any such f we obtain an induced homomorphism