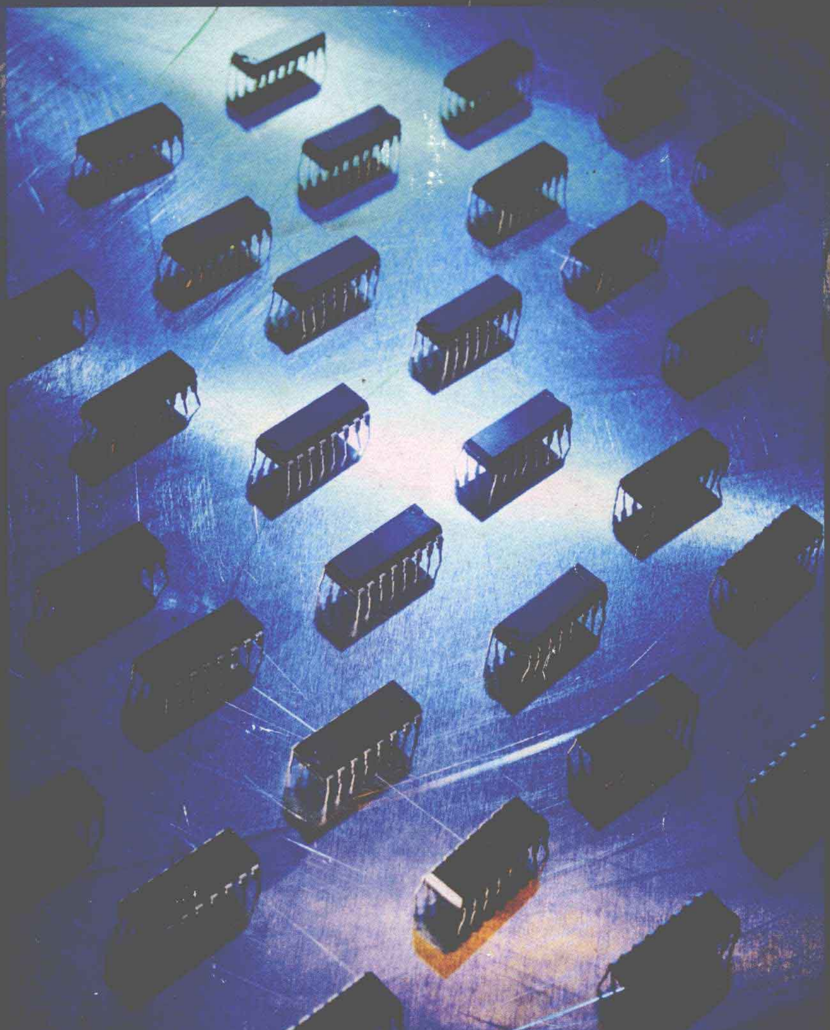
# Mathematics for the Liberal Arts

## MGF 1106

*Second Edition*



Miami-Dade Community College

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## LESSON 1 BASICS OF SETS

**Purpose of Lesson:** ***To develop a intuitive understanding of the concept of set.***

The concept of set, as it is known today, was developed at the end of the eighteenth century by the mathematician Georg Cantor (1845 – 1918). In his work, Cantor gave a general definition of what a set is and left himself open to attacks from those who were not too fond of his work. One of the well known paradoxes that this broad definition gave rise to was found by the logician Bertrand Russel (1872 – 1979), and although the terminology used is technical, the following example embodies its spirit:

Suppose that in a town there is a barber who shaves those who do not shave themselves. Does the barber shave himself or not?

In terms of sets, you can think that we have two sets: the set of all those who shave themselves and the set of all of those who do not. Ask yourself whether the barber can belong to these sets. If he belongs to the set of those who don't shave themselves, then he shaves himself because he shaves those who do not shave themselves. But now he belongs to the set of those who *do* shave themselves, a contradiction. Similarly, if we assume he shaves himself, then he doesn't, because he shaves only those who do not shave themselves.

This illustrates the complexities of set theory, but for our purposes we only need the language of sets and some elementary properties. We will take the naïve approach and develop an intuitive notion of sets. We will presume that sets will provide us with a useful way of analyzing logic and probability problems we will do in future lessons.

So let us begin with a very loose definition:

**Definition 1.1** *A set is a collection of objects that may be either concrete or abstract in nature and the objects that comprise the set are called its elements.*

**Example 1.1**

a) The collection composed of all persons who are taking this class.

b) The collection of all textbooks that can be purchased at your schools bookstore.

c) The collection of all natural numbers less that 20.

d) The collection of all stars in the universe.

e) The collection of all *natural numbers.*

f) The collection of all *real numbers*.

Although we can always describe a set using plain English, as was done in the example above, it is useful to introduce mathematical notation. Sets will be denoted by capital letters such as $A$, $B$, $C$, $X$, and $Y$. The elements of a set will be denoted by lower case letters such as $a, b, c, s$, and $y$. It is also convenient to introduce a way of describing a set. One obvious way is to simply list its elements, which is fairly easy if the set is *small* (that is, it has *few* elements) ; however, if the set is *large*, such as the set of all *natural numbers*, a list is not only impractical but also impossible. The notation that we will use is as follows: If the set is small and we can list its elements, then we will use $\{\ldots\}$ to group them. If the set is too large for a listing, then we will use $\{x \mid x$ has certain property$\}$. This is called *set builder notation* the symbol | reads "such that".

**Example 1.2**

The set of all ~~syllables~~ vowels can be described by $\{a, e, i, o, u\}$. If we call this set $S$, then we would write $S = \{a, e, i, o, u\}$.

**Example 1.3**

Let $E$ denote the set of all *even natural numbers* less than 10. This set can be described by $E = \{2, 4, 6, 8\}$.

**Example 1.4**

The elements of a set can also be sets: $\{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is an example of a set whose elements are themselves sets; namely $\{1\}, \{1, 2\}, \{1, 2, 3\}$.

**Example 1.5**

The set of all natural numbers cannot be listed because we can never get to the last element. If we use the letter **N** to denote this set, then we may describe it as follows:

$$N = \{1, 2, 3, 4, 5, \ldots\ldots\ldots\ldots\}$$

This, however, is a dangerous practice because there may be ambiguity as to what the next term will be. A better way to describe this set would be:

$$N = \{1, 2, 3, 4, 5, \ldots n, n + 1 \ldots\ldots\ldots\}$$

The two terms $n, n + 1$ tell the reader how to keep generating the elements of the set.

**Example 1.6**

The set of all even numbers may be described by $\boldsymbol{E} = \{2, 4, 6, \ldots\ldots\ldots\ldots\}$ but again it might not be clear what the next term should be. So we can instead write

$$\boldsymbol{E} = \{2n \mid n \text{ is a natural number}\}$$

This tells us exactly what the elements of the set are: they are all the multiples of 2 which is what all even numbers are.

**Definition 1.2** *When an object x belongs to a set A, we use the notation* $x \in A$. *If, on the on the hand, an object x is not an element of a set A, we use the notation* $x \notin A$.

**Example 1.7**

*a*) Let $S = \{1, 3, 4, 6, 7, 9\}$. Then we can write $3 \in S$ and $0 \notin S$.

*b*) Let **E** = $\{2n \mid n$ is an even number$\}$. Then we can write $4 \in S$ and $11 \notin S$.

*c*) Let $S = \{\{a\}, \{a, b\}, \{a, b, c\}\}$. Then we have that $\{a\} \in$ S, $\{a, b\}$ S, and $\{a, b, c\} \in S$. Note however that $a \notin S$ since the elements of S are sets themselves. A similar statement can be made about *b* and *c*. → a ∈ b ∉ S.

When we speak of sets, we usually have a particular collection of objects in mind. We call this collection our *universe;* all sets spoken of hereafter are composed of elements of this universe.

**Definition 1.3** The *set that forms our universe is called the universal set. The particular application one has in mind decides what this set is. We use the symbol U for the universal set.*

**Example 1.8**

If we are going speak of numbers, one possible universal set we can use is ***N***, the set of natural numbers. If this set should prove to be too small, we might want to use for our universal set the set of all *rational numbers* (i.e. all fractions whose numerator and denominator are *integers* and where the denominator is not 0). In the first case, we can let $U = \{1, 2, 3, 4, \ldots, n, n + 1, \ldots\}$ and in the latter case, $U = \{a/b \mid a$ and $b$ integers, $b \neq 0\}$.

In addition to universal sets, there is another important set called the *empty set.*

**Definition 1.4** *The empty set is the set that has no elements and is denoted by the symbol* $\varnothing$ *or simply* $\{ \}$.

**Example 1.9**

*a*) Consider the set of all human beings who are alive and whose age exceeds 200 years. I think it is safe to agree that this set is empty.

*b*) Let $X = \{x \mid x$ is a natural number and $2x = 1\}$. From algebra you probably know that the solution of this equation is $x = ½$, however, ½ is not a natural number so this set is empty; there is no natural number $x$ with the property that $2x = 1$.
Thus, we can write $X = \varnothing$ or $X = \{ \}$.

**Definition 1.5** *If every element of a set A is also an element of a set B, we call A a subset of B and write* $A \subseteq B$. *If, on the other hand, A is not a subset of B, we write* $A \not\subseteq B$.

Notice that this definition implies the *every set is a subset of itself* : For all sets $X$, $X \subseteq X$.

In addition, this definition also implies that if $A$ is any set, then $\varnothing \subseteq A$. We will have to wait until we learn some informal logic to see why this must be so.

**Example 1.10**

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $B = \{2, 4, 5\}$, and $C = \{0, 5, 6, 7\}$. Then we can write $B \subseteq A$ and $C \not\subset A$. We can also write $A \not\subset B$.

The next definition tells us what it means for two sets to be equal:

> **Definition 1.6** *Two sets A and B are equal if each is a subset of the other; $A = B$ if both $A \subseteq B$ and $B \subseteq A$.*

**Example 1.11** Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 1, 2, 5\}$, and $C = \{1, 3, 4, 5, 6\}$. Then we can say that $A = B$, since *the order in which elements are listed is not important.* We can also write $C \not\subseteq B$ and that $C \subseteq A$. ✓ $B \subseteq A$, $A \not\subseteq B$

> **Definition 1.6** *If every element of a set A is also an element of a set B and $A \neq B$, then we call A a proper subset of B and write $A \subset B$.*

In the example above, $C$ is a proper subset of $A$ and, of course, $B$.

**Example 1.12** Let $A = \{a, b, c, d, e, f\}$, $B = \{c, d, e, a, b, f\}$, and $C = \{a, c, b, d, f\}$. Then we can say that $A = B$, since *the order in which elements are listed is not important.* We can also write $C \subseteq B$ and $C \subseteq A$.

**Example 1.13** List all the subsets of the set $\{a, b, c, d\}$.

**Solution** We may organize our search as follows: first look for all subsets with no elements. Then look for those that have exactly one element; then for those with two elements; followed by f those with three elements, and finally for those with four elements. Keep in mind that a *subset of a set is itself a set all whose elements can be found in the original set.*

| # of elements | Sets | # of subsets |
|---|---|---|
| 0 | $\varnothing$ | 1 |
| 1 | $\{a\}, \{b\}, \{c\}, \{d\}$ | 4 |
| 2 | $\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$ | 6 |
| 3 | $\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$ | 4 |
| 4 | $\{a,b,c,d\}$ | 1 |
| | total | 16 |

So $\{a, b, c, d\}$ has 16 subsets.
Later, when we study probability, we will develop more powerful counting methods to find this number in a much easier way besides listing the actual subsets.
It will be found that:

> *The number of subsets of a set with n elements is $2^n$.*

You may have observed that we can immediately tell if two sets are not equal by simply comparing the number of elements each processes, but what if the sets are *infinite*?

Perhaps we should first understand what a finite set is:

**Definition 1.7** *We say a set A is finite if its elements can be paired in a one-to-one basis with the elements of the set {1, 2, 3, 4, …,n} for some integer n. We then say A has n elements. We also call n the cardinality of A and denote it by n(A).*

**Example 1.14** The set $A$ = {John, Jack, Joe, James} is finite because its elements can be paired of on a one-to-one basis with the elements of the set $B = \{1, 2, 3, 4\}$ like this:

| | | | | |
|---|---|---|---|---|
| *A*: | John | Jack | Joe | James |
| | ↕ | ↕ | ↕ | ↕ |
| *B*: | 1 | 2 | 3 | 4 |

We thus say that $A$ has 4 elements and we write $n(A) = 4$.

**Example 1.15** Since the empty set has no elements, we have that $n(\varnothing) = 0$.

We are now in the position to define:

**Definition 1.8** *We say a set A is infinite if it is not finite. In other words, no integer n can match the elements of our set with {1,2,3,4,5,…n}*

Notice that a set may be very large (for example the set of all stars on the known universe}, but still finite. To say that a set is infinite is a very strong statement and sheer large numbers is just not enough.

The idea behind "counting" the elements in a set is extremely important in mathematics and has been fertile soil for many interesting developments. To see just how strange sets may behave, consider the set of all positive integers **N** = {1, 2, 3, 4, 5, 6, 7 ,8, 9,…….} and the set of all even integers **E** = {2, 4, 6, 8, 10,……..}, both of which are infinite (convince yourself of this).

Any sensible way of comparing these two sets as to their sizes would call for some kind of matching procedure similar to the one in example 1.9 whereby an element of one set is assigned to an element of the other. Something like this:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **N**: | 1 | 2 | 3 | 4 | 6 ………… | $n$ | $n + 1$ …. |
| | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| **E**: | 2 | 4 | 6 | 8 | 12 ……….. | $2n$ | $2(n + 1)$ … |

It is surprising to note that, although set **E** is a *proper* subset of set **N**, it appears that they have the same "number of elements". How can this be? It is precisely questions like this one that led Cantor and others to study sets and their properties. In this course however, we will be dealing mostly with finite sets so we need not worry about the complexities of the infinite. After we introduce some set relations we will develop methods for computing the cardinality of more complicated sets.

## EXERCISES LESSON 1

***In each of the following exercises, answer the question as indicated.***

1. Write the first 10 elements of the set of *natural numbers*.
   N= {1,2,3,4,5,6,7,8,9,10}
2. Which of the following is true for the sets $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   $B = \{2, 3, 6, 7, 9\}$, and $C = \{6, 7, 2, 3, 9\}$, and $D = \{\{a\}, \{a, b\}, \{a,b,c\}, \varnothing\}$
   $E = \{a, b, c,\}$

   a. $A \subseteq B$ F
   ✓b. $B \subseteq A$ T
   c. $A \in B$ F
   d. $8 \subseteq B$ F
   e. $7 \subseteq C$ F
   ✓f. $9 \in A$ T
   ✓g. $8 \notin C$ T
   h. $a \in D$ F
   i. $\{a\} \subseteq D$ F
   j. $E \subseteq D$ F
   k. $\{a\} \in E$ F
   ✓l. $\{a\} \subseteq E$ T

3. How would you describe the set of all odd numbers using set builder notation?
4. List the first 5 *prime numbers*. 2, 3, 5, 7, 11
5. What is the set of all prime numbers divisible by 2? 2
6. Which of the following sets is empty?

   a) $\{x \mid x$ is an integer and $2x = 5\}$.
   b) $\{x \mid x$ *is a real number and* $x^2 = -4\}$.
   c) ~~The set of all prime numbers divisible by 2.~~
   d) The set of all odd numbers divisible by 2.

7. What is the cardinality of each of the following?

   a) {2, 3, 5, 6, 7, 9} 6
   b) {a, e, i, o, u} 5
   c) The set of all letters in the English alphabet. 26

8. Let $S = \{\{a\}, \{a, b, c\}, \{a, e, i\}, \{i,o,u\}\}$. What is the cardinality of $S$? 4

9. List the number of subsets of the sets {1, 2} and {1,2,3}. Combined with example 1.13, can you see the pattern in the numbers emerge?

## LESSON 2: SET OPERATIONS

| |
|---|
| **Purpose of Lesson:** ***To introduce the basic set operations of union, intersection, and complement formation and their related Venn Diagrams*** |

Just as with numbers, it is desirable and useful to be able to combine sets in different ways. The three basic operations with sets are *union*, *intersection*, and *complement formation*, that We will now define.

| |
|---|
| **Definition 2.1** *If U denotes the universal set and A and B are subsets of U, then their union is the set composed of all elements that belong to A, or B or both. This set is denoted by* $A \cup B$. |

Note the key word ***or***. It means in one, the other, or possibly both sets.

**Example 2.1** Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 9, 10\}$, and $C = \{a, b, c, d\}$. Then the union $A \cup B$ is given by the set $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$ and we can write:

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$$

Similarly, $A \cup C = \{1, 2, 3, 4, 5, 6, a, b, c, d\}$. Note that order is not important so we could just as easily write $A \cup C = \{a, b, 1, 3, 2, 4, 5, c, 6, d\}$, for this is the same set.

Now the intersection of two sets:

**Example 2.2** Given the sets of example 2.1, find $(A \cup B) \cup C$

**Solution** One can show that the order in which the set operations are performed is not important, but we will do what is being asked: first find $(A \cup B)$ and then find the union of the resulting set with $C$. We can accomplish this as follows:

$$(\{1, 2, 3, 4, 5, 6\} \cup \{2, 4, 6, 8, 9, 10\}) \cup \{a, b, c, d\}$$

$$= \{1, 2, 3,4, 5, 6, 8, 9, 10\} \cup \{a, b, c, d\}$$

$$= \{1, 2, 3, 4, 5, 6, 8, 9, 10, a, b, c, d\}$$

| |
|---|
| **Definition 2.2** *if U denotes the universal set and A and B are subsets of U, then their intersection is the set composed of all elements from A, and B. This set is denoted by* $A \cap B$, *that is, the set of all elements common to A and B. In the event that sets A and B have no elements in common, we say A and B are disjoint and their intersection is the empty set.* |

Note the key word ***and***.

**Example 2.3** Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 9, 10\}$, $C = \{a, b, c, d\}$, and $D = \{7, 8, 9\}$. Then the intersection $A \cap B$ is given by the set $\{2, 4, 6,\}$ and we can write:

$$A \cap B = \{2, 4, 6,\}$$

Similarly, $A \cap D = \varnothing$ and $B \cap C = \varnothing$ since sets $A$ and $D$ have no elements in common and neither do sets $B$ and $C$.

**Example 2.4** We can also combine both operations: Let $R = \{a, e, i, o, u\}$, $S = \{a, b, c, d, e\}$, and $T = \{e, f, g, h\}$. Find $(R \cup S) \cap T$.

**Solution** We first find $(R \cup S)$:

$$(R \cup S) = \{a, e, i, o, u, b, c, d\}$$

Now we intersect this with $T$,

$$\{a, e, i, o, u, b, c, d\} \cap \{e, f, g, h\} = \{e\}$$

Thus, $(R \cup S) \cap T = \{e\}$, a set of a single element.

It will also be useful introduce the concept of all elements that are *not on a set*; this we call the *complement* of a set.

> **Definition 2.3** *if $U$ denotes the universal set and $A$ a subset of $U$, then the complement of $A$, Denoted by $A'$, is the set of all elements in $U$ but not in $A$.*

**Example 2.5** Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{2, 3, 4, 7\}$. The complement of $A$ is given by $A' = \{1, 5, 6, 8, 9\}$. You see that this set is *the set of all elements of $U$ not in $A$.*

You may want to form $A'$ by crossing out from $U$ the elements of $A$; what remains is $A'$:

$$A' = \{1, \not{2}, \not{3}, \not{4}, 5, 6, \not{7}, 8, 9\} = \{1, 5, 6, 8, 9\}.$$

**Example 2.6** If U = {John, Jack, Jill, Jenny, Joe}, A = {John, Jill}, and B = {Jenny, Joe}, find $A' \cup B'$.

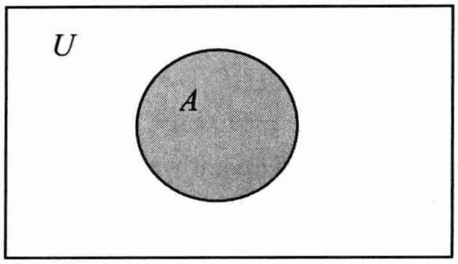
**Solution First**, we must find the complements of A and B respectively; then we must find the union of these complements.

$A'$ = {Jack, Jenny, Joe}

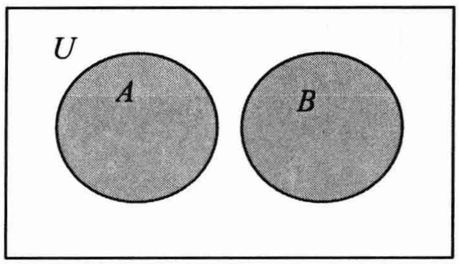
$B'$ = {John, Jack, Jill}

So $A' \cup B'$ = {Jack, Jenny, Joe, John, Jill}

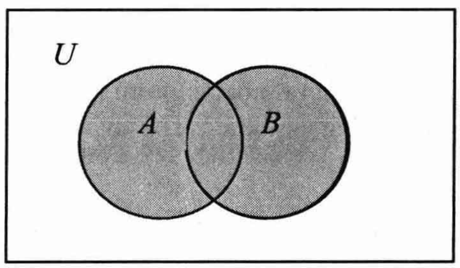
Set operations and relations may be visualized by using *Venn Diagrams*. The list below illustrates the basic relations (note that $U$ stands for the universal set).



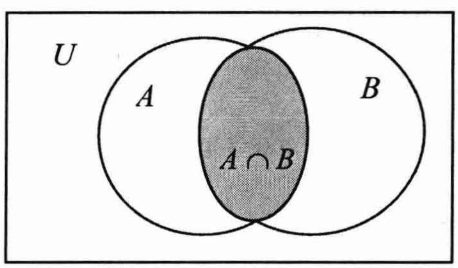

The shaded region shows set $A$



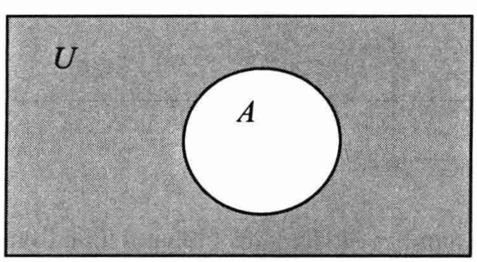

The shaded region shows the fact that sets $A$ and $B$ are disjoint.




The shaded region shows the union of sets $A$ and $B$




The shaded region shows the intersection of sets $A$ and $B$



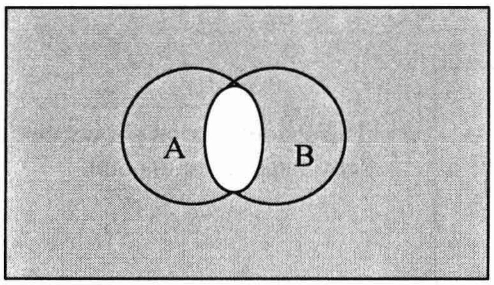

The shaded region shows the complement of $A$

With these basic Venn diagrams, we may form more complicated ones as the example on the next page illustrates.

**Example 2.7 Shade** the region corresponding to $(A \cap B)'$. Assume A and B have non-empty intersection.

**Solution**

What we have here is the complement of the intersection. To find, it we must shade everything not in the intersection, so we leave the intersection alone.




The shaded region shows the complement of the intersection

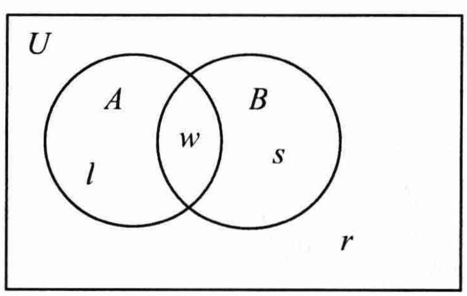
We will use these diagrams when we study logic and probability.

Now lets go back to the concept of cardinality. Suppose we are given two sets, $A$ and $B$ and suppose we know something about $n(A)$ and $n(B)$. Can we obtain from these the values of $n(A \cup B)$ and $n(A \cap B)$, or some relation involving all these? The answer is yes as the following analysis shows:

Again, we limit ourselves to finite sets, but the arguments can be extended to infinite ones.

Certainly if A and B are disjoint, $n(A \cup B) = n(A) + n(B)$, for this just says the total number of elements in the union of two disjoint sets is given by the sum of the elements of one set and the elements of the other. Similarly, $n(A \cap B) = 0$, for if two sets are disjoint, their intersection is empty and $n(\varnothing) = 0$.

Now suppose that A and B are not disjoint. Consider the Venn diagram below,




The lowercase letters in region represent the number of elements in that region from which it follows that $n(A \cup B) = l + w + s$. Note the $l$ represents the number of elements in $A$ but not in $B$; $w$ represents the number of elements in the intersection of $A$ and $B$, and $s$ represents the number of elements in $B$ but not in $A$. Now let us do a little algebra and write

$$n(A \cup B) = l + w + s + w - w \quad \text{(we added and subtracted } w\text{, so nothing has changed)}$$

$$n(A \cup B) = (l + w) + (s + w) - w \qquad \text{(associative property of addition)}$$

Finally, note that the terms in the parentheses represent nothing but $n(A)$ and $n(B)$, respectively, and $w$ is nothing but $n(A \cap B)$ from which it follows that:

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

**This is an extremely important formula so keep it in mind.**

Notice that this formula reduces to the familiar $n(A \cup B) = n(A) + n(B)$ when $A$ and $B$ are disjoint, for then $A \cap B = \varnothing$ and $n(A \cap B) = 0$.

Let us see how we use this formula.

**Example 2.8** Suppose that set $A$ has 10 elements, set $B$ has 6 elements and we find that $A \cap B$ has 3 elements. How many elements does $A \cup B$ have?

**Solution** Using our formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have that:

$$n(A \cup B) = 10 + 6 - 3 = 13$$

so the union has 13 elements.

**Example 2.9** We found that in a group of ten persons, six read *Time* magazine and eight read *Forbes* magazine. How many read both magazines?

**Solution** You might have already concluded that the answer is 4, but let us see how our formula produces this result; let $T$ be the set of persons who read *Time* magazine and $F$ be the set who read *Forbes* magazine. Since this conforms our universe, $T \cup F$ must have 10 elements (the persons in the survey). Using our formula, we have:

$$n(T \cup F) = n(T) + n(F) - n(T \cap F)$$

$$10 = 6 + 8 - n(T \cap F)$$

or

$$10 = 14 - n(T \cap F)$$

Solving for $n(T \cap F)$ we get $n(T \cap F) = 4$.

**Exercises Lesson 2**

***PART A*: *Answer each of the following questions as indicated.***

1. If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $B = \{3, 4, 5, 7\}$, and $C = \{8, 11, 12, 15\}$, then find:

   a. $A \cup B$
   b. $A \cup C$
   c. $(A \cup B) \cup C$
   d. $B \cup C$
   e. $A \cup (C \cup B)$

2. If $U = \{a, e, i, o, u\}$, $A = \{a, i, o\}$, $B = \{a, o, u\}$, and $C = \{i, o, u\}$, find:

   a. $(A \cup B)'$
   b. $A' \cap B'$
   c. $(A \cup B) \cap C'$
   d. How do the results of parts *a* and *b* compare?

3. Given that *A* has 10 elements, *B* has 14 elements, and there are 4 elements common to *A* and *B*, how many elements are there in $A \cup B$?

4. Draw a Venn diagram of $A' \cup B$.

5. Using the operations of union, intersection and complement formation, describe the region below. (Note, there might be more than one way of doing it).