SCIENCE SURVEYS: 1

Analysis and Synthesis of Linear Time-Variable Systems

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PREFACE

The material in this monograph is based on a dissertation of the same title that was submitted in partial fulfillment of the requirements for a Ph.D. in engineering at the University of California, Los Angeles. The research was supported in part by grant AFOSR 62–68, contract AF 33(651)-7154.

This monograph is an attempt to present a complete synthesis technique for linear time-variable systems, an area of research that has not been receiving sufficient attention in recent years. The analysis problem is treated briefly in chapter 3. The reader is assumed to have a knowledge of the fundamental properties of linear ordinary differential equations.

I wish to thank Professor C. T. Leondes of UCLA for his advice and encouragement while this work was being done, Dr. E. B. Stear with whom I worked on the material in chapter 3, and finally my wife, May, for her help and understanding.

A. R. S.

Los Angeles, April, 1964

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Chapter 1

INTRODUCTION

The analysis and synthesis of linear time-invariant systems have, to a great extent, dominated the efforts of engineers in all fields. As a result, a large body of literature on these subjects presently exists at the expense of linear time-variable and nonlinear systems. The reason for this domination stems, first, from the fact that general techniques for the analysis, and consequently the synthesis, of linear time-invariant systems are relatively simple compared to those for linear time-variable and nonlinear systems. Secondly, many linear time-variable and nonlinear systems can be adequately approximated (in some sense) by a linear time-invariant system.

The purpose of this monograph is to partially fill some of the voids in the areas of analysis and synthesis of linear time-variable systems. In particular, the results are slanted toward the field of feedback control systems; they are not, however, limited to this area. The problem of analysis is considered in some detail (chap. 3), but the primary concern is with the problem of synthesis.

It might be added that the techniques and ideas have been developed in enough detail that they are valuable from a practical standpoint. Many of the problems that are most likely to be encountered in a practical design situation have been examined. As a result, a large class of linear time-variable systems can be synthesized by means of the techniques developed in this monograph.

Defining the System

Any process that produces a response (output) when an excitation (input) is applied to it, can be called a system (the term transmittance is also used). A system may be depicted as a block as in figure 1.1 where W represents the system,

Figure 1.1. A general system.

$x(t)$ represents the input, $y(t)$ the output, and t is the independent variable. If, in addition, the system W is linear, then it must satisfy the following definition:

DEFINITION.

If 1. *an input* x_1 *produces an output* y_1,
2. *an input* x_2 *produces an output* y_2, *and*
3. *an input* $c_1x_1 + c_2x_2$ *produces an output* $c_1y_1 + c_2y_2$, *where* x_1, x_2, y_1, *and* y_2 *are arbitrary functions of* t *and* c_1 *and* c_2 *are arbitrary constants, then* W *is a linear system.*

The definition includes a wide variety of systems, for example, distributed-parameter, lumped-parameter, constant-coefficient, and variable-coefficient. This monograph is limited to the investigation of the class of linear systems that can be described by an ordinary linear differential equation of the form

$$\sum_{i=0}^{n} a_i(t) \frac{d^i y}{dt^i} = \sum_{i=0}^{n} b_i(t) \frac{d^i x}{dt^i}, \qquad \tau \leqq t < +\infty,$$
$$\left. \frac{d^i y}{dt^i} \right|_{t=\tau} = 0, \qquad i = 0, 1, 2, \cdots, n-1, \tag{1.1}$$

where the independent variable t is time, τ is the time of application of the input, the $a_i(t)$ and $b_i(t)$ are continuous and deterministic functions of time, $y(t)$ is the output of the system, and $x(t)$ is the input. In addition, $a_n(t)$ is assumed to be unity (without loss of generality), and any or all of the $b_i(t)$ may be zero in a particular case. This requires that the order of the operator (the upper limit on the summation)

$$\sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i}$$

is always equal to, or greater than, the order of the operator

$$\sum_{i=0}^{n} b_i(t) \frac{d^i}{dt^i}.$$

In a physical system this means that there are no net differentiations between the input and the output, which is a valid constraint from a physical standpoint (see chap. 4). In the following text, the operator

$$\sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i}$$

will be referred to as the integral operator of equation (1.1), since it indicates that an integral operation must be performed on the input to generate the output. Similarly, the operator

$$\sum_{i=0}^{n} b_i(t) \frac{d^i}{dt^i}$$

will be referred to as the differential operator, since it indicates that a differential operation is performed on the input in the system.

Another method of defining the types of systems investigated in this monograph is by means of their weighting functions. A linear differential equation of the form in equation (1.1) has associated with it a weighting function (unit impulse response function) $W(t, \tau)$ of the form (refs. [3], [16], [21])

$$\begin{aligned} W(t, \tau) &= \sum_{j=1}^{n} \beta_j(\tau) q_j(t) + b_n(t)\delta(t - \tau), \qquad t \geqq \tau, \\ &= 0, \qquad t < \tau, \end{aligned} \tag{1.2}$$

where $\delta(t - \tau)$ is the Dirac delta function, the $q_j(t)$ are linearly independent solutions of the homogeneous portion of equation (1.1), that is,

$$\sum_{i=0}^{n} a_i(t) \frac{d_i q_j}{dt^i} = 0, \quad j = 1, 2, \cdots, n, \tag{1.3}$$

and the $\beta_j(\tau)$ are given by

$$\beta_j(\tau) = \sum_{i=0}^{n} (-1)^i \frac{d^i}{d\tau^i} [b_i(\tau)\alpha_j(\tau)], \tag{1.4}$$

in which the $\alpha_i(\tau)$ are determined from the set of simultaneous equations

$$\begin{aligned} \alpha_1(\tau)q_1(\tau) + \cdots + \alpha_n(\tau)q_n(\tau) &= 0, \\ \alpha_1(\tau) \frac{dq_1(\tau)}{d\tau} + \cdots + \alpha_n(\tau) \frac{dq_n(\tau)}{d\tau} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_1(\tau) \frac{d^{n-1}q_1(\tau)}{d\tau^{n-1}} + \cdots + \alpha_n(\tau) \frac{d^{n-1}q_n(\tau)}{d\tau^{n-1}} &= 1. \end{aligned} \tag{1.5}$$

The condition that $W(t, \tau)$ be zero for $t < \tau$ guarantees that the system is physically realizable. The equivalence of a weighting function of the form in equation (1.2) with a differential equation of the form in (1.1) is discussed in Appendix I. It is shown that, subject to the necessity of an adequate number of derivatives of the $\beta_j(\tau)$ and $q_j(t)$, a differential equation of the form in equation (1.1) can be generated from a weighting function of the form in equation (1.2). Because of this equivalence, either equation (1.1) or (1.2) may be used to define a particular system.

Previous Work

The previous work in the analysis and synthesis of linear time-variable systems in engineering applications may be conveniently classified according to the mathematical representations, called transfer functions, used to describe the systems. Evaluation of this work must depend a great deal upon the various properties of these mathematical representations.

In evaluating the representations, three important criteria are these:

1. How difficult is it to arrive at the representation?
2. How much information about the system response is *readily* available from each representation?
3. How difficult are the operations of combining (cascade and parallel combinations, etc.) and manipulating systems when they are described by the representation?

The three most general types of transfer functions are the following:

1. Weighting functions.
2. Time-variable frequency response functions.
3. Differential equations.

Techniques for generating integral transforms for general linear time-variable differential equations have also been developed [1], [14]; however, these techniques have proved to be too specialized or too difficult for use in a general theory that includes both analysis and synthesis.

Weighting Functions (refs. [3], [5], [13], [15], [18], [20], [21], [30])

The weighting function has long been used as a tool in the analysis of linear systems. Its value lies in the fact that knowledge of the weighting function allows the response of a system to be determined for any input by means of the convolution integral; that is, if $W(t, \tau)$ is the weighting function of a linear system, then for an input $x(t)$ the output $y(t)$ is given by

$$y(t) = \int_{-\infty}^{t} W(t, \tau)x(\tau)\, d\tau. \tag{1.6}$$

Thus from the standpoint of criterion (2), the weighting function is an excellent method of representation.

Problems associated with the weighting function are that it is very difficult to determine for a given general linear time-variable differential equation, and that usually it is not expressible in a closed form. In both analysis and synthesis these difficulties may rule out the weighting function as a useful representation. Actually, the amount of difficulty associated with determining a weighting function depends upon the form of the differential equation.

To answer criterion (3), the three critical operations of combination of systems will be examined. These operations are the following:

1. Parallel combination of systems.
2. Cascade combination of systems.
3. Finding an inverse system.

A parallel combination of systems is illustrated in figure 1.2. Let the system W_1 have the weighting function $W_1(t, \tau)$ and the system W_2 have

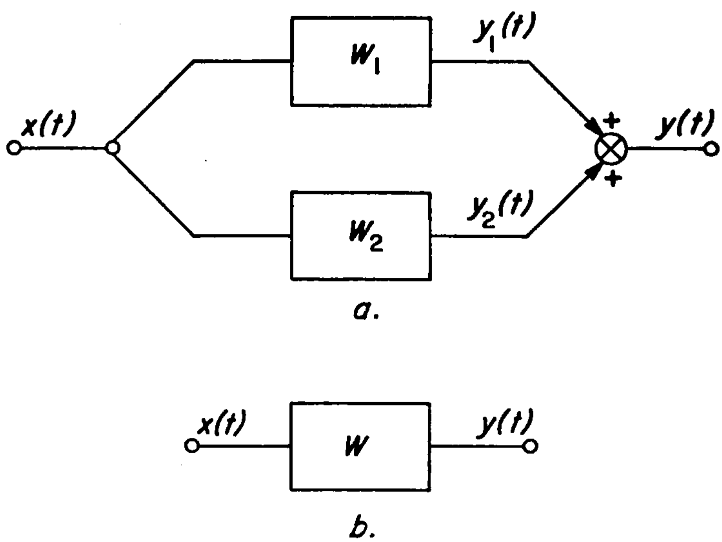

Figure 1.2. Parallel combination of systems.

the weighting function $W_2(t, \tau)$; then

$$y_1(t) = \int_{-\infty}^{t} W_1(t, \tau)x(\tau)d\tau,$$

$$y_2(t) = \int_{-\infty}^{t} W_2(t, \tau)x(\tau)\, d\tau, \tag{1.7}$$

and

$$y(t) = y_1(t) + y_2(t) = \int_{-\infty}^{t} [W_1(t, \tau) + W_2(t, \tau)]x(\tau)\, d\tau.$$

Then the system W in figure 1.2, *b*, which is equivalent to figure 1.2, *a*, has a weighting function $W(t, \tau)$ given by

$$W(t, \tau) = W_1(t, \tau) + W_2(t, \tau). \tag{1.8}$$

Obviously if $y(t) = y_1(t) - y_2(t)$ then the equivalent system has the weighting function

$$W(t, \tau) = W_1(t, \tau) - W_2(t, \tau). \tag{1.9}$$

A cascade combination of systems is illustrated in figure 1.3. Again

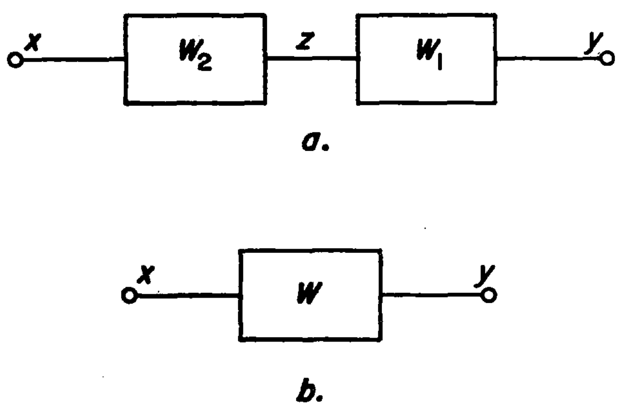

Figure 1.3. Cascade combination of systems.

let W_1 have the weighting function $W_1(t, \tau)$, and let W_2 have the weighting function $W_2(t, \tau)$. Then

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} W_1(t, \theta)z(\theta)\, d\theta, \\ z(t) &= \int_{-\infty}^{t} W_2(t, \tau)x(\tau)\, d\tau; \end{aligned} \tag{1.10}$$

combining these equations,

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} d\theta W_1(t, \theta) \int_{-\infty}^{\theta} W_2(\theta, \tau)x(\tau)\, d\tau \\ &= \int_{-\infty}^{t} d\tau\, x(\tau) \int_{\tau}^{t} W_1(t, \theta) W_2(\theta, \tau) d\theta. \end{aligned} \tag{1.11}$$

The equivalent system W (fig. 1.3, *b*) then has the weighting function

$$W(t, \tau) = \int_{\tau}^{t} W_1(t, \theta) W_2(\theta, \tau) d\theta. \tag{1.12}$$

The final necessary operation, the finding of an inverse system, is illustrated in figure 1.4. Two systems are inverse to each other if, when

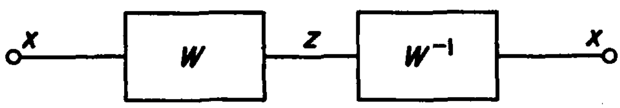

Figure 1.4. Inverse systems.

$x(t)$ is applied as an input to the cascade combination of these systems, the output is also $x(t)$. The system equivalent to this cascade combination evidently has as its weighting function a Dirac delta function. This is apparent from the convolution integral

$$y(t) = \int_{-\infty}^{t} \delta(t - \tau) x(\tau)\, d\tau = x(t). \tag{1.13}$$

Then by virtue of equation (1.12) the weighting function of the inverse system, $W^{-1}(t, \tau)$, is the solution of the integral equation

$$\delta(t - \tau) = \int_{\tau}^{t} W^{-1}(t, \theta) W(\theta, \tau)\, d\theta, \tag{1.14}$$

in which $W(t, \tau)$ is the weighting function of W.

The three necessary operations for combining systems represented by weighting functions are then defined by equations (1.8), (1.12), and (1.14). The first two operations are simple and straightforward, but the third—the finding of an inverse—is generally difficult. In addition, the solution may be in the form of an infinite series, which will prove to be too unwieldy to manipulate in the synthesis problem, and the solution must therefore be approximated, usually by truncating the series.

Mal'chikov [18] and Gladkov [13] have examined the problem of synthesizing a given time-variable weighting function as a feedback system using modified versions of these techniques of combination. The disadvantage of their scheme is the requirement that an integral equation must always be solved. Cruz and Van Valkenberg [8], by synthesizing linear time-variable systems in an open-loop configuration, do not encounter the problem of finding inverses.

Time-Variable Frequency Response Functions

The concept of the time-variable frequency function representation of a linear time-variable system was introduced by Zadeh [27], [28], [29]. Because of its close relationship to the system weighting function, its properties are, from the standpoint of the three criteria above, equivalent to those of the weighting function.

The time-variable frequency response function, which is denoted $H(j\omega, t)$, is related to the weighting function by the pair of integrals [30]:

$$H(j\omega, t) = \int_{-\infty}^{\infty} W(t, \tau)e^{-j\omega(t-\tau)}\, d\tau, \tag{1.15}$$

$$W(t, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega, t)e^{j\omega(t-\tau)}\, d\omega. \tag{1.16}$$

Equation (1.15) may be considered the definition of $H(j\omega, t)$.

The amount of work involved in determining $H(j\omega, t)$ from a differential equation is equal to that necessary for determining $W(t, \tau)$. The output $y(t)$ of the system for a particular input $x(t)$ can be determined from the integral

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega, t)X(j\omega)e^{j\omega t}\, d\omega, \tag{1.17}$$

where $X(j\omega)$ is the Fourier transform of $x(t)$. Apparently $H(j\omega, t)$ contains as much readily available information as the weighting function and, in general, more than the differential equation.

The techniques for combining time-variable frequency response functions can be obtained immediately by applying the definition in (1.15) to equations (1.8), (1.12), and (1.14). For combining $H_1(j\omega, t)$ and $H_2(j\omega, t)$ in parallel, equation (1.8) becomes

$$H(j\omega, t) = H_1(j\omega, t) + H_2(j\omega, t). \tag{1.18}$$

For combining $H_1(j\omega, t)$ and $H_2(j\omega, t)$ in cascade, equation (1.12) becomes

$$H(j\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_1(j\omega + j\omega', t)e^{j\omega' t} \cdot \left\{\int_{-\infty}^{\infty} H_2(j\omega, \theta)e^{-j\omega'\theta}\, d\theta\right\} d\omega'. \tag{1.19}$$

Finally, the inverse of $H(j\omega, t)$ can be determined by solving the integral equation

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^{-1}(j\omega + j\omega', t) e^{j\omega' t} \cdot \left\{ \int_{-\infty}^{\infty} H(j\omega, \theta) e^{-j\omega'\theta} \, d\theta \right\} d\omega', \tag{1.20}$$

which is the equivalent of equation (1.14). Obviously the same difficulty encountered in determining inverses for weighting functions is present in determining inverses for time-variable frequency response functions.

Engineers have found little use for the time-variable frequency response function beyond the analysis of some linear time-variable electrical networks. They have done almost nothing with it in the area of synthesis of systems, since in the choosing of an overall response function the correlation between the system output and the response function is not readily apparent, as it is, for instance, in the case of weighting functions.

Differential Equations

Differential equations are the most common form of representation for a physical system, for the reason that physical laws (such as Ohm's law and Newton's law) when applied to a particular system produce as governing equations, differential equations. To determine the response of the system, these equations must then be solved. Obtaining the solution of a general linear time-variable differential equation is difficult; therefore, a differential equation generally contains less readily available information about system response than a weighting function or time-variable frequency response function. From the standpoint of the first two criteria above, differential equations rate high with respect to the first and low with respect to the second.

The techniques for combining differential equations are developed in chapter 2. With respect to the third criterion, it will be seen that differential equations rate very high. Previous work in the area of techniques for combining differential equations is limited to a few papers by Darlington [9]–[11], in which he briefly mentions that such techniques might be used, but he neither develops nor makes use of them.

In this discussion some of the advantages and disadvantages of the three general representations of linear time-variable systems have been discussed. The discussion is not to imply that one representation should be used to the exclusion of the others. By judicious use of all three, if they are readily available, there may be a great saving in time in analyzing and synthesizing such systems.

Scope of the Monograph

The purpose of this monograph is to develop a general technique for analyzing and, in particular, for synthesizing linear time-variable systems. The emphasis is on techniques for synthesizing feedback control systems, although the techniques need not be confined to these systems.

The first step in the development is the discussion, in chapter 2, of an algebra of differential equations that allows for the combination of differential equations in a manner not unlike the combination of matrices. In chapter 3 the algebra is applied to the analysis of linear systems via signal flow graph theory. In chapter 4 the algebra is applied to the problem of synthesizing given weighting functions (or differential equations) as feedback systems. Chapter 5 is devoted to the producing of overall system functions. In chapter 6 a criterion for determining the reducibility of a linear system is developed along with a technique for reducing the order of a reducible system. Finally, in chapter 7, techniques for approximating given differential equations are developed. The monograph thus provides a complete technique for sythesizing linear time-variable systems and provides, to a lesser degree, techniques for the analysis of these systems.

Chapter 2

AN OPERATOR ALGEBRA FOR DIFFERENTIAL EQUATIONS

In chapter 1 three representations of linear transmittances were discussed along with the advantages and disadvantages of each. It was there stated that the main advantages of the differential equation representation are the relative ease of manipulation and the straightforward techniques of the combining of differential equations.

In this chapter the techniques for combining and the rules of manipulation of differential equations are developed in the form of an algebra of linear transformations.

The Necessary Operations

Since a linear differential equation of the form

$$\sum_{i=0}^{n} a_i(t) \frac{d^i y}{dt^i} = \sum_{j=0}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{2.1}$$

is a linear transformation of x into y, the algebra is an algebra of linear transformations [2]. The transformations (linear differential equations) that are considered will all have the form of equation (2.1), where some of the $a_i(t)$ and $b_j(t)$ may be zero. In the following development, capital letters (A, B, C, $\cdots$) are used to represent differential equations of the form of equation (2.1). In addition, all initial conditions are assumed to be zero.

The algebra of differential equations involves three operations:

1. Addition of two differential equations; that is,

$$A + B = C. \tag{2.2}$$

2. Multiplication of a differential equation by a scalar; that is,

$$Ap(t) = B, \tag{2.3}$$

or

$$p(t)A = C. \tag{2.4}$$

(Clearly, if $p(t)$ is a constant, $B = C$.)

3. Multiplication of two differential equations; that is,

$$BA = C. \tag{2.5}$$

These operations are indicated in block-diagram form in figure 2.1. Obviously, in order for the algebra to be useful, each of the operations must be defined. In addition, the following useful properties of the algebra will be defined:

1. A unity element.
2. A zero element.
3. An additive inverse.
4. A multiplicative inverse.

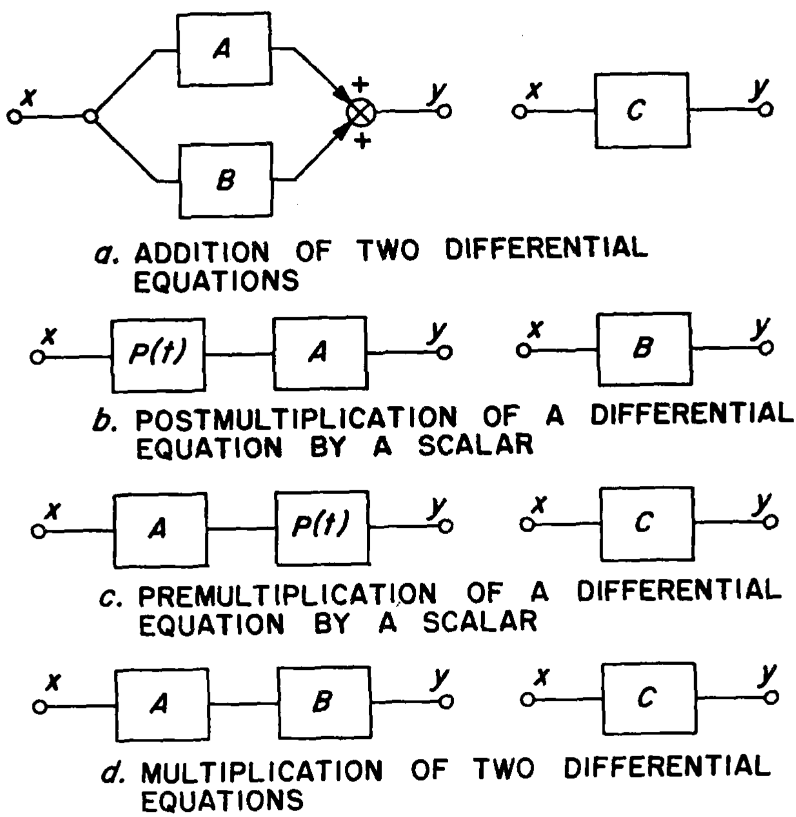

Figure 2.1. Operations of transformation algebra.

MULTIPLICATION OF TWO DIFFERENTIAL EQUATIONS

It is necessary to define multiplication first since it is used in defining addition.

Any differential equation of the form in equation (2.1) can be divided into two parts—a differential operator and an integral operator (fig. 2.2).

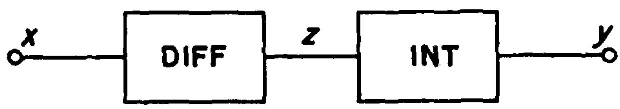

Figure 2.2. Block diagram of differential equation in terms of the notation of equation (2.1).

The relationships between the variables x, y, and z are, in terms of the notation of equation (2.1),

$$z = \sum_{j=0}^{n} b_j(t) \frac{d^j x}{dt^j}, \tag{2.6}$$

$$\sum_{i=0}^{n} a_i(t) \frac{d^i y}{dt^i} = z. \tag{2.7}$$

The multiplication of two differential equations can be represented by figure 2.3, *a*. The equations that define the relationships between the

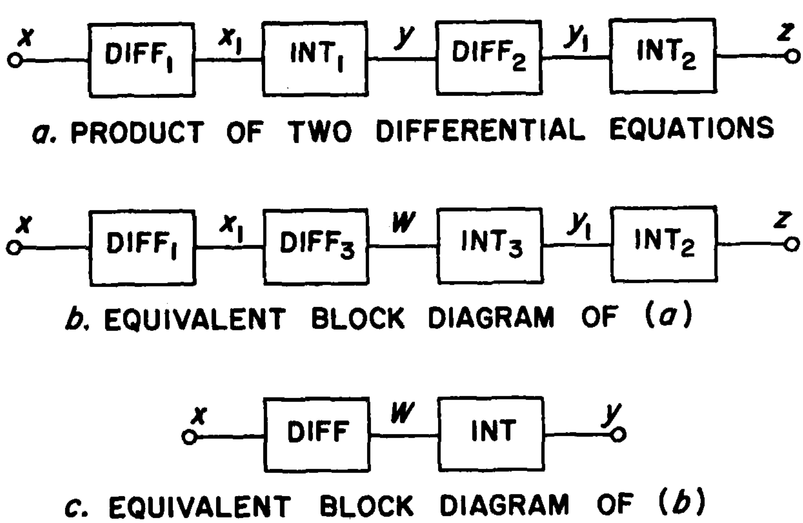

Figure 2.3. Multiplication of two differential equations