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High Efficient and Accuracy Numerical Methods for Optimal Control Problems

Chen Yanping（陈艳萍） Lu Zuliang（鲁祖亮）

（最优控制问题高效高精度算法）

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Preface to the Series in Information and Computational Science

Since the 1970s, Science Press has published more than thirty volumes in its series Monographs in Computational Methods. This series was established and led by the late academician, Feng Kang, the founding director of the Computing Center of the Chinese Academy of Sciences. The monograph series has provided timely information of the frontier directions and latest research results in computational mathematics. It has had great impact on young scientists and the entire research community, and has played a very important role in the development of computational mathematics in China.

To cope with these new scientific developments, the Ministry of Education of the People's Republic of China in 1998 combined several subjects, such as computational mathematics, numerical algorithms, information science, and operations research and optimal control, into a new discipline called Information and Computational Science. As a result, Science Press also reorganized the editorial board of the monograph series and changed its name to Series in Information and Computational Science. The first editorial board meeting was held in Beijing in September 2004, and it discussed the new objectives, and the directions and contents of the new monograph series.

The aim of the new series is to present the state of the art in Information and Computational Science to senior undergraduate and graduate students, as well as to scientists working in these fields. Hence, the series will provide concrete and systematic expositions of the advances in information and computational science, encompassing also related interdisciplinary developments.

I would like to thank the previous editorial board members and assistants, and all the mathematicians who have contributed significantly to the monograph series on Computational Methods. As a result of their contributions the monograph series achieved an outstanding reputation in the community. I sincerely wish that we will extend this support to the new Series in Information and Computational Science, so that the new series can equally enhance the scientific development in information and computational science in this century.

Shi Zhongci

2005.7

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Chapter 1

Introduction

Optimal control problems have been extensively utilized in many aspects of the modern life such as social, economic, scientific and engineering numerical simulation. Due to the wide application of these problems, they must be solved successfully with efficient numerical methods. Among these numerical methods, finite element discretization of the state equation was widely applied though other methods were also used.

Many researchers have made a lot of works on some topics of finite element methods for optimal control problems. In particular, for optimal control problem governed by linear elliptic state equations, there were two early papers on the numerical approximation for linear-quadratic control-constrained problems by Falk [1] and Geveci [2]. More recently, Arnautu and Neittaanmäki [3] contributed further error estimates to this class of problems. Moreover, we refer to Casas [4] who proves convergence results for optimal control problems governed by linear elliptic equations with control in the coefficient. Most recently, Meyer and Rösch have studied the superconvergence property for linear-quadratic optimal control problem, they also investigated the L^∞ estimates with standard finite element for this problem in [5]. Liu and Yan [6, 7] have derived a posteriori error estimates for finite element approximation of convex optimal control problems and boundary control problems respectively. For optimal control problem governed by linear parabolic state equations, a priori error estimates of finite element approximation were studied in, for example [8] and [9]. A posteriori error estimates for this problem were discussed by Liu and Yan [10].

Mixed finite element methods were much more important methods for a certain class of problems which contains the gradient of the state variable in the objective functional. Thus the accuracy of gradient was important in numerical approximation of the state equations. When it comes to these problems, mixed finite element methods should be used with which both the scalar variable and its flux variable can be approximated in the same accuracy. Although mixed finite element methods were extensively used in engineering numerical simulations, it has not been fully used in computational optimal control problems yet. Particularly, there doesn't seem much work on theoretical analysis of mixed finite element approximation of optimal control problem in the literature although there were some works of the mixed finite element methods for elliptic equation and parabolic equation, for example, see [11-17].

In numerical analysis, a superconvergent method was one which converges faster than generally expected. For example in the finite element method approximation to Poisson's equation in two dimensions, using piecewise linear elements, the average error in the gradient was first order. However under certain conditions it was possible to recover the gradient at certain locations within each element to second order. Superconvergence of finite element approximations for optimal control problems have been extensive studies for standard finite element methods and mixed finite element methods. In [18], Meyer and Rösch constructed a postprocessing projection operator and derived a quadratic superconvergence of the control by finite element methods. In [19], Liu and Yan considered recovery type superconvergence and a posteriori error estimates for control problem governed by Stokes equations. Next, Yan [20] analyzed the superconvergence property of finite element method for an optimal control problem governed by integral equations. A priori error estimates and superconvergence for an optimal control problem of bilinear type were obtained in [21]. Compared with standard finite element methods, the mixed finite element methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. In [22], we used the postprocessing projection operator, which was defined by Meyer and Rösch [18] to prove a quadratic superconvergence of the control problems by mixed finite element methods. We derived error estimates and superconvergence of mixed methods for convex optimal control problems in [23].

The finite volume element method was a discretization technique for partial differential equations. Due to its local conservative property and other attractive properties such as the robustness with the unstructured meshes, the finite volume element method was widely used in computational fluid dynamics. In general, two different functional spaces (one for the trial space and one for the test space) were used in the finite volume element method. Owing to the two different spaces, the numerical analysis of the finite volume element method was more difficult than that of the finite element method and finite difference method. Since the method was proposed, there has been many results in the literature. Early work for the finite volume element method can be found in [24,25]. In [26], Bank and Rose obtained the result that the finite volume approximation was comparable with the finite element approximation in H^1-norm. The optimal L^2-error estimate was obtained in [27] under the assumption that $f \in H^1$. In [28], the authors also obtained the H^1-norm and maximum-norm error estimates. In [29], Chatzipantelidis proposed a nonconforming finite volume element method and obtained the L^2-norm and H^1-norm error estimates. Recently, Chou and Ye proposed a discontinuous finite volume element method. Unified error analysis for conforming, nonconforming and discontinuous finite volume element method was presented in [30]. High order finite volume element method can be found in, e.g., [31,32].

Recently, Hinze introduced a variational discretization concept for optimal control problems in [33]. Its key feature is not to discretize the control but to implicitly utilize the optimality conditions and the discretization of the state and the co-state for the discretization of the control. It can not only save computational cost but also improve the order of convergence of the control variable. In [34], Hinze and Meyer discussed the variational discretization methods of Lavrentiev-regularized state constrained elliptic optimal control problems. The authors studied the variational discretization methods for optimal control problems governed by convection dominated diffusion equations in [35].

The spectral method employed global polynomials as the trial functions for the discretization of partial differential equations. It provided very accurate approximations with a relatively small number of unknowns when the solutions were smooth. Recently, the spectral method has been extended to approximate an unconstrained optimal control problem, see, for example, [36]. In [37], the flow optimal control was successfully approximated by the Legendre-Galerkin spectral method, where both the unconstrained and the constrained cases were discussed. In [38], spectral method was used to approximate state constrained control problems governed by the first bi-harmonic equation. However, the spectral accuracy generally cannot be achieved when the approximated solutions have lower regularities, and this was typically the case when, for example, there existed the control constraints in optimal control problems (the so-called constrained optimal control problems). Thus, the spectral method was not widely used in solving constrained distributed optimal control problems where the solutions often have singularities at the boundary of constraints even though all the initial data was smooth. Although there has much work on the finite element method for numerically solving constrained optimal control problems, and on the mixed finite element method for the optimal control problems. It seems that there was no much work on the spectral method for the optimal control problems. Furthermore, the optimality conditions, which were normally the starting point of spectral approximation, were just partial differential equations systems for unconstrained optimal control problems, while those for constrained optimal control problems contain variational inequalities. This also arised new issues in analyzing and solving the systems discretised using the spectral method.

The book consists of nine chapters. In the first chapter, we presented the introduction of the book. The Chapter 2 gave some preliminaries including Sobolev spaces, the basic concepts of finite element methods, mixed finite element methods, and optimal control problems. The Chapter 3 discussed finite element methods for some optimal control problems, for example, elliptic optimal control problems, parabolic optimal control problems, and optimal control problems with oscillating coefficients. We considered a priori error estimates of mixed finite element methods for elliptic optimal control problems, parabolic optimal control problems, hyperbolic optimal control problems, fourth order optimal control problems, and nonlinear optimal control problems in Chapter 4. Next we discussed a posteriori error estimates of mixed finite element methods for optimal control problems in Chapter 5 and superconvergence of mixed finite element methods for optimal control problems in Chapter 6. In Chapter 7, we introduced the finite volume element methods for optimal control problems. In Chapter 8, we considered the variational discretization methods for optimal control problems. Finally, we considered the Legendre-Galerkin spectral methods for optimal control problems in Chapter 9. In this book, Yanping Chen was mainly responsible for the Chapters 1-5 and Zuliang Lu was responsible for the Chapters 6-9.

The subject of our book was computational mathematics and numerical analysis. The emphases and selection of the topics reflected our involvement in the field about the past 10 years. In this book, there were many different methods for optimal control problems, including finite element methods, mixed finite element methods, finite volume element methods, Legendre-Galerkin spectral methods, and so on. To our best knowledge, this was among the first book to introduce many different numerical methods for optimal control problems. At the same time, illustrations and tables were very clearness and beautiful. In our opinion, the book should be especially suitable for students who undertake research related to computational issues of optimal control problems.

We would like to thank our collaborators and students for their suggestions and helps. We also thank the Science Press for their efforts to make the publishing of the book.

Chapter 2

Some preliminaries

In this chapter, the general basic theory of Sobolev spaces, the finite element methods and optimal control problems will be provided. We introduce some definitions, notations and a few well-known properties and conclusions. More details can be found in [39-41].

2.1 Sobolev spaces

Let Ω be bounded open sets in $\mathbb{R}^d$ ($d = 1, 2,$ or 3) with a Lipschitz boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and the semi-norm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : \ w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$).

We denote by $L^s(0, T; W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0, T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))} = (\int_0^T \|v\|_{W^{m,q}(\Omega)}^s dt)^{\frac{1}{s}}$ for $s \in [0, \infty)$ and the standard modification for $s = \infty$. Similarly, one define the spaces $H^1(0, T; W^{m,q}(\Omega))$ and $C^l(0, T; W^{m,q}(\Omega))$.

We set $0 = t_0 < t_1 < \cdots < t_N = T$, N is a positive integer, $k_i = t_i - t_{i-1}$, $i = 1, 2, \cdots, N$, $k = \max\limits_{i\in[1,N]} \{k_i\}$. Let $g^i = g(x, t_i)$, we define for $1 \leqslant p < \infty$ the discrete time-dependent norms

$$|||g(x,t)|||_{L^p(J;H^s(\Omega))} := \left(\sum_{i=1-l}^{N-l} k_{i+l} \left\| g^i \right\|_s^p \right)^{\frac{1}{p}},$$

where $l = 0$ for the control variable $u(x, t)$ and the state variable $y(x, t)$ and $l = 1$ for the co-state variable $p(x, t)$, with the standard modification for $p = \infty$.

We will introduce the following the trace theorem.

Lemma 2.1.1 *Given $\phi \in H^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain* [42]*, there exists a* *constant C depending only on Ω such that*

$$\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)} \leqslant C\|\phi\|_{1,\Omega}. \tag{2.1}$$

In particular,

$$\|\phi\|_{L^2(\partial\Omega)} \leqslant C\|\phi\|_{1,\Omega}. \tag{2.2}$$

Moreover, if $g \in H^{\frac{1}{2}}(\partial\Omega)$, there exists $\phi \in H^1(\Omega)$ such that $\phi|_{\partial\Omega} = g$, and

$$\|\phi\|_{1,\Omega} \leqslant C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{2.3}$$

Next we introduce the Friedrichs-Poincaré's inequality [43].

Lemma 2.1.2 *Assume that Ω is a Lipschitz domain, there exists a constant C depending only on Ω such that, for any $f \in \tilde{H}^1(\Omega)$,*

$$\|f\|_{L^2(\Omega)} \leqslant C\|\nabla f\|_{L^2(\Omega)}. \tag{2.4}$$

Now, we introduce some standard results for mixed finite element methods.

Lemma 2.1.3 *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Given $f \in L^2(\Omega)$, then there exists $\boldsymbol{v} \in H^1(\Omega) \times H^1(\Omega)$ such that*

$$\mathrm{div}\boldsymbol{v} = f, \quad in\ \ \Omega, \tag{2.5}$$

and

$$\|\boldsymbol{v}\|_{1,\Omega} \leqslant C\|f\|_{L^2(\Omega)}. \tag{2.6}$$

Proof Let $B \in \mathbb{R}^2$ be a ball containing Ω, g be a zero extension of f from Ω into the ball B, and ϕ be the solution of the boundary problem

$$\begin{aligned} \Delta\phi &= g, \quad \text{in } B,\\ \phi &= 0, \quad \text{on } \partial B. \end{aligned}$$

It is known that ϕ satisfies the following a priori estimate

$$\|\phi\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}, \tag{2.7}$$

and therefore $\boldsymbol{v} = \nabla\phi$ satisfies (2.5) and (2.6). □

A usual technique to obtain error estimates for finite element approximations is to work in a reference element and then change variables to prove results for a general element. Let us introduce some notations and recall some basic estimates.

Fix a reference $\hat{\tau} \in \mathbb{R}^2$. Given a simple $\tau \in \mathbb{R}^2$, there exists an invertible map $F : \hat{\tau} \to \tau$, $F(\hat{\tau}) = A\hat{\tau} + b$, with $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$. We call h_τ the diameter of τ and ρ_τ the diameter of the largest ball inscribed in τ. We will use the regularity assumption on the elements, namely, many of our estimates will depend on a constant σ such that

$$\frac{h_\tau}{\rho_\tau} \leqslant \sigma.$$

It is known that for the matrix norm associated with the Euclidean vector norm, the following estimates hold:

$$||A|| \leqslant \frac{h_\tau}{\rho_{\hat{\tau}}},$$

and

$$||A^{-1}|| \leqslant \frac{h_{\hat{\tau}}}{\rho_\tau}.$$

With any $\phi \in L^2(\tau)$ we associate $\hat{\phi} \in L^2(\hat{\tau})$ in the usual way, namely,

$$\phi(x) = \hat{\phi}(\hat{x}),$$

where $x = F(\hat{x})$.

Now, we recalling the so-called inverse estimates which are a fundamental tool in finite element analysis.

Lemma 2.1.4 *Given a simplex T there exists a constant $C = C(\sigma, k, \hat{T})$ such that, for any $p \in P_k(T)$,*

$$||\nabla p||_{L^2(T)} \leqslant \frac{C}{h_T} ||p||_{L^2(T)}. \tag{2.8}$$

Proof Since $P_k(\hat{T})$ is a finite dimensional space, all the norms defined on it are equivalent. In particular, there exists a constant $\hat{C}$ depending on k and $\hat{T}$ such that

$$||\hat{\nabla}\hat{p}||_{L^2(\hat{T})} \leqslant \hat{C} ||\hat{p}||_{L^2(\hat{T})}, \tag{2.9}$$

for any $\hat{p} \in P_k(\hat{T})$. An easy computation shows that

$$\nabla p = A^{-\mathrm{T}} \hat{\nabla} \hat{p},$$

where $A^{-\mathrm{T}}$ is the transpose matrix of A^{-1}. Therefore, using the bound for $||A^{-1}||$ and together with (2.9) we have

$$\begin{aligned}
&\int_T |\nabla p|^2 dx \\
&= \int_{\hat{T}} |A^{-\mathrm{T}} \hat{\nabla} \hat{p}|^2 |\mathrm{det} A| d\hat{x} \leqslant ||A^{-1}|| \int_{\hat{T}} |\hat{\nabla} \hat{p}|^2 |\mathrm{det} A| d\hat{x} \\
&\leqslant \hat{C} \frac{h_{\hat{T}}^2}{\rho_T^2} \int_{\hat{T}} |\hat{p}|^2 |\mathrm{det} A| d\hat{x} = \hat{C} \frac{h_{\hat{T}}^2}{\rho_T^2} \int_T |p|^2 dx \leqslant \hat{C} \sigma^2 \frac{h_{\hat{T}}^2}{h_T^2} \int_T |p|^2 dx,
\end{aligned}$$

and the proof is complete. □

The following lemmas are very important in deriving a posteriori error estimates.

Lemma 2.1.5 *Let $\hat{\pi}_h$ be the average interpolation operator defined in* [44]. *For $m = 0$ or 1, $1 \leqslant q \leqslant \infty$ and $\forall\, v \in W^{1,q}(\Omega^h)$,*

$$|v - \hat{\pi}_h v|_{W^{m,q}(\tau)} \leqslant \sum_{\bar{\tau}' \cap \bar{\tau} \neq \varnothing} C h_\tau^{1-m} |v|_{W^{1,q}(\tau')}. \tag{2.10}$$

Lemma 2.1.6 *Let π_h be the standard Lagrange interpolation operator* [41]. *Then for $m = 0$ or 1, $1 < q \leqslant \infty$ and $\forall\, v \in W^{2,q}(\Omega^h)$,*

$$|v - \pi_h v|_{W^{m,q}(\tau)} \leqslant C h_\tau^{2-m} |v|_{W^{2,q}(\tau)}. \tag{2.11}$$

Lemma 2.1.7 *For all $v \in W^{1,q}(\Omega^h)$, $1 \leqslant q < \infty$,*

$$\|v\|_{W^{0,q}(\partial\tau)} \leqslant C \left(h_\tau^{-\frac{1}{q}} \|v\|_{W^{0,q}(\tau)} + h_\tau^{1-\frac{1}{q}} |v|_{W^{1,q}(\tau)} \right).$$

For parabolic problems, let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t^n = n\Delta t$, $n \in \mathbb{Z}$, $\psi^n = \psi^n(x) = \psi(x, t^n)$, we introduce the following standard $L^2(\Omega)$-orthogonal projection $Q_h : U \to U_h$ [45], which for all $\psi \in X$ satisfies

$$(\psi^n - Q_h\psi^n, v_h) = 0, \quad \forall\, v_h \in U_h, \tag{2.12}$$

and elliptic projection operator $R_h : W \to W_h$, which for any $\phi \in V$ satisfies

$$a(\phi^n - R_h\phi^n, w_h) = (A(x, t_n)\nabla(\phi^n - R_h\phi^n), \nabla w_h) = 0, \quad \forall\, w_h \in W_h. \tag{2.13}$$

We have the following approximation properties:

$$\|\psi^n - Q_h\psi^n\|_{-s} \leqslant C h^{1+s} |\psi^n|_1, \quad \forall\, \psi^n \in H^1(\Omega),\ s = 0, 1, \tag{2.14}$$

$$\|\phi^n - R_h\phi^n\| \leqslant C h^2 \|\phi^n\|_2, \quad \forall\, \phi^n \in H^2(\Omega). \tag{2.15}$$

More details can be found in Subsection 4.2.

2.2 Finite element methods for elliptic equations

Over the last decades, the finite element method which was introduced by engineers in the 1960s, has become the most important numerical method for partial differential equations, particularly for equations of elliptic and parabolic types. This method is based on the variational form of the boundary value problem and approximates the exact solution by a piecewise polynomial function. It is more easily adapted to the geometry of the underlying domain than the finite difference method, and for symmetric positive definite elliptic problems it reduces to a finite linear system with a symmetric positive definite matrix.