Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

Linear Algebra and Its Applications

fourth edition

Gilbert Strang
Massachusetts Institute of Technology

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Preface

Revising this textbook has been a special challenge, for a very nice reason. So many people have read this book, and taught from it, and even loved it. The spirit of the book could never change. This text was written to help our teaching of linear algebra keep up with the enormous importance of this subject—which just continues to grow.

One step was certainly possible and desirable—*to add new problems*. Teaching for all these years required hundreds of new exam questions (especially with quizzes going onto the web). I think you will approve of the extended choice of problems. The questions are still a mixture of *explain* and *compute*—the two complementary approaches to learning this beautiful subject.

I personally believe that many more people need linear algebra than calculus. Isaac Newton might not agree ! But he isn't teaching mathematics in the 21st century (and maybe he wasn't a great teacher, but we will give him the benefit of the doubt). Certainly the laws of physics are well expressed by differential equations. Newton needed calculus—quite right. But the scope of science and engineering and management (and life) is now so much wider, and linear algebra has moved into a central place.

May I say a little more, because many universities have not yet adjusted the balance toward linear algebra. Working with curved lines and curved surfaces, the first step is always to *linearize*. Replace the curve by its tangent line, fit the surface by a plane, and the problem becomes linear. The power of this subject comes when you have ten variables, or 1000 variables, instead of two.

You might think I am exaggerating to use the word "beautiful" for a basic course in mathematics. Not at all. This subject begins with two vectors v and w, pointing in different directions. The key step is to *take their linear combinations*. We multiply to get $3v$ and $4w$, and we add to get the particular combination $3v + 4w$. That new vector is in the *same plane* as v and w. When we take all combinations, we are filling in the whole plane. If I draw v and w on this page, their combinations $cv + dw$ fill the page (and beyond), but they *don't go up* from the page.

In the language of linear equations, I can solve $cv + dw = b$ exactly when the vector b lies in the same plane as v and w.

Matrices

I will keep going a little more to convert combinations of three-dimensional vectors into linear algebra. If the vectors are $v = (1, 2, 3)$ and $w = (1, 3, 4)$, put them into the **columns of a matrix**:

$$\textbf{matrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

To find combinations of those columns, "**multiply**" the matrix by a vector (c, d):

$$\textbf{Linear combinations } \boldsymbol{cv + dw} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Those combinations fill a *vector space*. We call it the **column space** of the matrix. (For these two columns, that space is a plane.) To decide if $b = (2, 5, 7)$ is on that plane, we have three components to get right. So we have three equations to solve:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \quad \text{means} \quad \begin{aligned} c + \ d &= 2 \\ 2c + 3d &= 5 \\ 3c + 4d &= 7. \end{aligned}$$

I leave the solution to you. The vector $b = (2, 5, 7)$ does lie in the plane of v and w. If the 7 changes to any other number, then b won't lie in the plane—it will *not* be a combination of v and w, and the three equations will have no solution.

Now I can describe the first part of the book, about linear equations $Ax = b$. The matrix A has n columns and m rows. *Linear algebra moves steadily to n vectors in m-dimensional space.* We still want combinations of the columns (in the column space). We still get m equations to produce b (one for each row). Those equations may or may not have a solution. They always have a least-squares solution.

The interplay of columns and rows is the heart of linear algebra. It's not totally easy, but it's not too hard. Here are four of the central ideas:

1. The **column space** (all combinations of the columns).
2. The **row space** (all combinations of the rows).
3. The **rank** (the number of independent columns) (or rows).
4. **Elimination** (the good way to find the rank of a matrix).

I will stop here, so you can start the course.

Web Pages

It may be helpful to mention the web pages connected to this book. So many messages come back with suggestions and encouragement, and I hope you will make free use of everything. You can directly access *http://web.mit.edu/18.06*, which is continually updated for the course that is taught every semester. Linear algebra is also on MIT's OpenCourseWare site *http://ocw.mit.edu*, where 18.06 became exceptional by including videos of the lectures (which you definitely don't have to watch...). Here is a part of what is available on the web:

1. Lecture schedule and current homeworks and exams with solutions.
2. The goals of the course, and conceptual questions.
3. Interactive Java demos (audio is now included for eigenvalues).
4. Linear Algebra Teaching Codes and MATLAB problems.
5. Videos of the complete course (taught in a real classroom).

The course page has become a valuable link to the class, and a resource for the students. I am very optimistic about the potential for graphics with sound. The bandwidth for voiceover is low, and FlashPlayer is freely available. This offers a *quick review* (with active experiment), and the full lectures can be downloaded. I hope professors and students worldwide will find these web pages helpful. My goal is to make this book as useful as possible with all the course material I can provide.

Other Supporting Materials

Student Solutions Manual 0-495-01325-0 The Student Solutions Manual provides solutions to the odd-numbered problems in the text.

Instructor's Solutions Manual 0-030-10568-4 The Instructor's Solutions Manual has teaching notes for each chapter and solutions to all of the problems in the text.

Structure of the Course

The two fundamental problems are $Ax = b$ and $Ax = \lambda x$ for square matrices A. The first problem $Ax = b$ has a solution when A has *independent columns*. The second problem $Ax = \lambda x$ looks for *independent eigenvectors*. A crucial part of this course is to learn what "independence" means.

I believe that most of us learn first from examples. You can see that

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \textbf{does \textit{not} have independent columns}.$$

Column 1 plus column 2 equals column 3. A wonderful theorem of linear algebra says that the three rows are not independent either. The third row must lie in the same plane as the first two rows. Some combination of rows 1 and 2 will produce row 3. You might find that combination quickly (I didn't). In the end I had to use elimination to discover that the right combination uses 2 times row 2, minus row 1.

Elimination is the simple and natural way to understand a matrix by producing a lot of zero entries. So the course starts there. But don't stay there too long! You have to get from combinations of the rows, to independence of the rows, to "dimension of the row space." That is a key goal, to see whole spaces of vectors: the ***row space*** and the ***column space*** and the ***nullspace.***

A further goal is to understand how the matrix *acts*. When A multiplies x it produces the new vector Ax. The whole space of vectors moves—it is "transformed" by A. Special transformations come from particular matrices, and those are the foundation stones of linear algebra: diagonal matrices, orthogonal matrices, triangular matrices, symmetric matrices.

The eigenvalues of those matrices are special too. I think 2 by 2 matrices provide terrific examples of the information that eigenvalues λ can give. Sections 5.1 and 5.2 are worth careful reading, to see how $Ax = \lambda x$ is useful. Here is a case in which small matrices allow tremendous insight.

Overall, the beauty of linear algebra is seen in so many different ways:

1. **Visualization.** Combinations of vectors. Spaces of vectors. Rotation and reflection and projection of vectors. Perpendicular vectors. Four fundamental subspaces.

2. Abstraction. Independence of vectors. Basis and dimension of a vector space. Linear transformations. Singular value decomposition and the best basis.

3. Computation. Elimination to produce zero entries. Gram–Schmidt to produce orthogonal vectors. Eigenvalues to solve differential and difference equations.

4. Applications. Least-squares solution when $Ax = b$ has too many equations. Difference equations approximating differential equations. Markov probability matrices (the basis for Google!). Orthogonal eigenvectors as principal axes (and more . . .).

To go further with those applications, may I mention the books published by Wellesley-Cambridge Press. They are all linear algebra in disguise, applied to signal processing and partial differential equations and scientific computing (and even GPS). If you look at *http://www.wellesleycambridge.com*, you will see part of the reason that linear algebra is so widely used.

After this preface, the book will speak for itself. You will see the spirit right away. The emphasis is on understanding—*I try to explain rather than to deduce*. This is a book about real mathematics, not endless drill. In class, I am constantly working with examples to teach what students need.

Acknowledgments

I enjoyed writing this book, and I certainly hope you enjoy reading it. A big part of the pleasure comes from working with friends. I had wonderful help from Brett Coonley and Cordula Robinson and Erin Maneri. They created the LaTeX files and drew all the figures. Without Brett's steady support I would never have completed this new edition.

Earlier help with the Teaching Codes came from Steven Lee and Cleve Moler. Those follow the steps described in the book; MATLAB and Maple and Mathematica are faster for large matrices. All can be used (*optionally*) in this course. I could have added "Factorization" to that list above, as a fifth avenue to the understanding of matrices:

[L,U,P] = lu(A)	for linear equations
[Q,R] = qr(A)	to make the columns orthogonal
[S,E] = eig(A)	to find eigenvectors and eigenvalues.

In giving thanks, I never forget the first dedication of this textbook, years ago. That was a special chance to thank my parents for so many unselfish gifts. Their example is an inspiration for my life.

And I thank the reader too, hoping you like this book.

Gilbert Strang

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Chapter

1 Matrices and Gaussian Elimination

1.1 INTRODUCTION

This book begins with the central problem of linear algebra: *solving linear equations*. The most important case, and the simplest, is when the number of unknowns equals the number of equations. We have ***n equations in n unknowns***, starting with $n = 2$:

$$\begin{array}{lcccccc} \textbf{Two equations} & 1x & + & 2y & = & 3 \\ \textbf{Two unknowns} & 4x & + & 5y & = & 6. \end{array} \tag{1}$$

The unknowns are x and y. I want to describe two ways, *elimination* and *determinants*, to solve these equations. Certainly x and y are determined by the numbers 1, 2, 3, 4, 5, 6. The question is how to use those six numbers to solve the system.

1. Elimination Subtract 4 times the first equation from the second equation. This eliminates x from the second equation, and it leaves one equation for y:

$$\textbf{(equation 2)} - \textbf{4(equation 1)} \qquad -3y = -6. \tag{2}$$

Immediately we know $y = 2$. Then x comes from the first equation $1x + 2y = 3$:

$$\textbf{Back-substitution} \qquad 1x + 2(2) = 3 \qquad \text{gives} \qquad x = -1. \tag{3}$$

Proceeding carefully, we check that x and y also solve the second equation. This should work and it does: 4 times $(x = -1)$ plus 5 times $(y = 2)$ equals 6.

2. Determinants The solution $y = 2$ depends completely on those six numbers in the equations. There must be a formula for y (and also x). It is a "ratio of determinants" and I hope you will allow me to write it down directly:

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2. \tag{4}$$

That could seem a little mysterious, unless you already know about 2 by 2 determinants. They gave the same answer $y = 2$, coming from the same ratio of -6 to -3. If we stay with determinants (which we don't plan to do), there will be a similar formula to compute the other unknown, x:

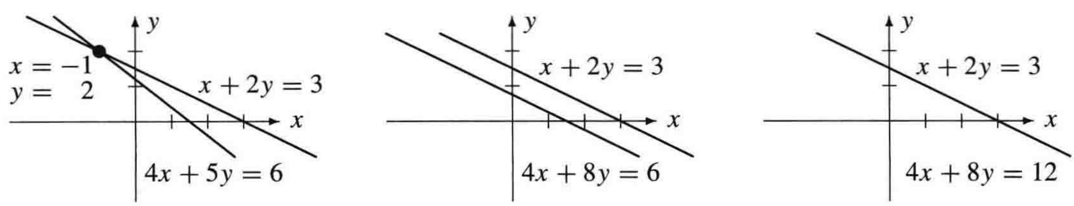
$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 2 \cdot 6}{1 \cdot 5 - 2 \cdot 4} = \frac{3}{-3} = -1. \tag{5}$$

Let me compare those two approaches, looking ahead to real problems when n is much larger ($n = 1000$ is a very moderate size in scientific computing). The truth is that direct use of the determinant formula for 1000 equations would be a total disaster. It would use the million numbers on the left sides correctly, but not efficiently. We will find that formula (Cramer's Rule) in Chapter 4, but we want a good method to solve 1000 equations in Chapter 1.

That good method is ***Gaussian Elimination***. This is the algorithm that is constantly used to solve large systems of equations. From the examples in a textbook ($n = 3$ is close to the upper limit on the patience of the author and reader) you might not see much difference. Equations (2) and (4) used essentially the same steps to find $y = 2$. Certainly x came faster by the back-substitution in equation (3) than the ratio in (5). For larger n there is absolutely no question. Elimination wins (and this is even the best way to compute determinants).

The idea of elimination is deceptively simple—you will master it after a few examples. It will become the basis for half of this book, simplifying a matrix so that we can understand it. Together with the mechanics of the algorithm, we want to explain four deeper aspects in this chapter. They are:

1. Linear equations lead to ***geometry of planes***. It is not easy to visualize a nine-dimensional plane in ten-dimensional space. It is harder to see ten of those planes, intersecting at the solution to ten equations—but somehow this is almost possible. Our example has two lines in Figure 1.1, meeting at the point $(x, y) = (-1, 2)$. Linear algebra moves that picture into ten dimensions, where the intuition has to imagine the geometry (and gets it right).
2. We move to ***matrix notation***, writing the n unknowns as a vector x and the n equations as $Ax = b$. We multiply A by "elimination matrices" to reach an upper triangular matrix U. Those steps factor A into L *times* U, where L is lower



One solution $(x, y) = (-1, 2)$ **Parallel: No solution** **Whole line of solutions**

Figure 1.1 The example has one solution. Singular cases have none or too many.

triangular. I will write down A and its factors for our example, and explain them at the right time:

$$\textbf{Factorization} \qquad A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \boldsymbol{L \text{ times } U}. \tag{6}$$

First we have to introduce matrices and vectors and the rules for multiplication. Every matrix has a ***transpose*** A^{T}. This matrix has an ***inverse*** A^{-1}.

3. In most cases elimination goes forward without difficulties. The matrix has an inverse and the system $Ax = b$ has one solution. In exceptional cases the method will *break down*—either the equations were written in the wrong order, which is easily fixed by exchanging them, or the equations don't have a unique solution.

That ***singular case*** will appear if 8 replaces 5 in our example:

$$\begin{array}{lrcrcl} \textbf{Singular case} & 1x & + & 2y & = & 3 \\ \textbf{Two parallel lines} & 4x & + & \mathbf{8}y & = & 6. \end{array} \tag{7}$$

Elimination still innocently subtracts 4 times the first equation from the second. But look at the result!

$$\textbf{(equation 2)} - \textbf{4(equation 1)} \qquad 0 = -6.$$

This singular case has ***no solution***. Other singular cases have ***infinitely many solutions***. (Change 6 to 12 in the example, and elimination will lead to $0 = 0$. Now y can have *any value*.) When elimination breaks down, we want to find every possible solution.

4. We need a rough count of the ***number of elimination steps*** required to solve a system of size n. The computing cost often determines the accuracy in the model. A hundred equations require a third of a million steps (multiplications and subtractions). The computer can do those quickly, but not many trillions. And already after a million steps, roundoff error could be significant. (Some problems are sensitive; others are not.) Without trying for full detail, we want to see large systems that arise in practice, and how they are actually solved.

The final result of this chapter will be an elimination algorithm that is about as efficient as possible. It is essentially the algorithm that is in constant use in a tremendous variety of applications. And at the same time, understanding it in terms of *matrices*—the coefficient matrix A, the matrices E for elimination and P for row exchanges, and the final factors L and U—is an essential foundation for the theory. I hope you will enjoy this book and this course.

1.2 THE GEOMETRY OF LINEAR EQUATIONS

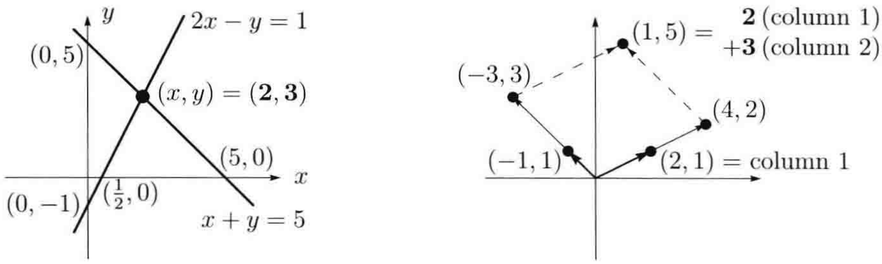
The way to understand this subject is by example. We begin with two extremely humble equations, recognizing that you could solve them without a course in linear algebra. Nevertheless I hope you will give Gauss a chance:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$

We can look at that system *by rows or by columns*. We want to see them both.

The first approach concentrates on the separate equations (the ***rows***). That is the most familiar, and in two dimensions we can do it quickly. The equation $2x - y = 1$ is represented by a *straight line* in the x-y plane. The line goes through the points $x = 1$, $y = 1$ and $x = \frac{1}{2}$, $y = 0$ (and also through (2, 3) and all intermediate points). The second equation $x + y = 5$ produces a second line (Figure 1.2a). Its slope is $dy/dx = -1$ and it crosses the first line at the solution.

The point of intersection lies on both lines. It is the only solution to both equations. That point $x = 2$ and $y = 3$ will soon be found by "elimination."



(a) **Lines meet at $x = 2, y = 3$** (b) **Columns combine with 2 and 3**

Figure 1.2 Row picture (two lines) and column picture (combine columns).

The second approach looks at the ***columns*** of the linear system. The two separate equations are really ***one vector equation***:

$$\textbf{Column form} \qquad x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}.$$

The problem is ***to find the combination of the column vectors on the left side that produces the vector on the right side.*** Those vectors (2, 1) and (−1, 1) are represented by the bold lines in Figure 1.2b. The unknowns are the numbers x and y that multiply the column vectors. The whole idea can be seen in that figure, where 2 times column 1 is added to 3 times column 2. Geometrically this produces a famous parallelogram. Algebraically it produces the correct vector (1, 5), on the right side of our equations. The column picture confirms that $x = 2$ and $y = 3$.

More time could be spent on that example, but I would rather move forward to $n = 3$. Three equations are still manageable, and they have much more variety:

$$\textbf{Three planes} \qquad \begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \qquad (1)$$

Again we can study the rows or the columns, and we start with the rows. Each equation describes a ***plane*** in three dimensions. The first plane is $2u + v + w = 5$, and it is sketched in Figure 1.3. It contains the points $(\frac{5}{2}, 0, 0)$ and (0, 5, 0) and (0, 0, 5). It is determined by any three of its points—provided they do not lie on a line.

Changing 5 to 10, the plane $2u + v + w = 10$ would be parallel to this one. It contains (5, 0, 0) and (0, 10, 0) and (0, 0, 10), twice as far from the origin—which is

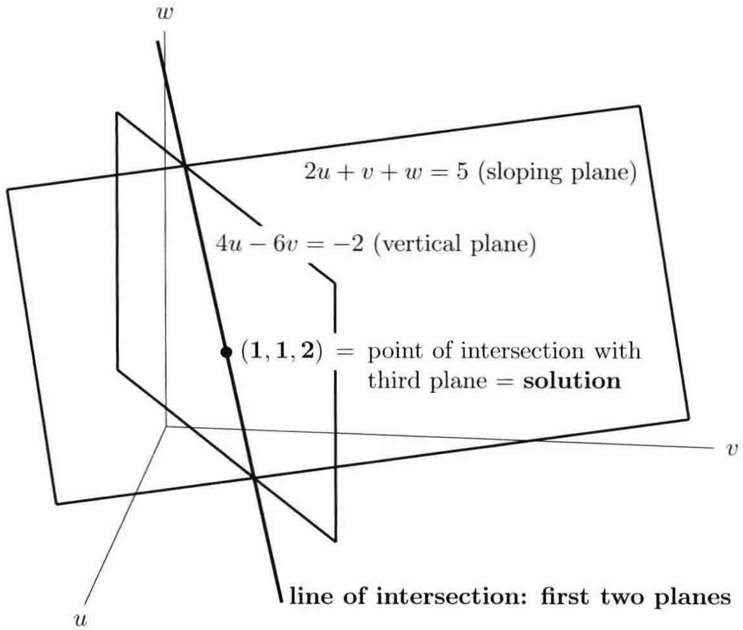


Figure 1.3 The row picture: three intersecting planes from three linear equations.

the center point $u = 0$, $v = 0$, $w = 0$. Changing the right side moves the plane parallel to itself, and the plane $2u + v + w = 0$ goes through the origin.

The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value. The coefficient of w is zero, but this remains a plane in 3-space. (The equation $4u = 3$, or even the extreme case $u = 0$, would still describe a plane.) The figure shows the intersection of the second plane with the first. That intersection is a line. *In three dimensions a line requires two equations*; in n dimensions it will require $n - 1$.

Finally the third plane intersects this line in a point. The plane (not drawn) represents the third equation $-2u + 7v + 2w = 9$, and it crosses the line at $u = 1$, $v = 1$, $w = 2$. That triple intersection point (1, 1, 2) solves the linear system.

How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a "plane." It is no longer a two-dimensional plane in 3-space; somehow it has "dimension" $n - 1$. It must be flat and extremely thin within n-dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane $t = 0$ cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary x-y plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary x-y plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n - 1)$-dimensional plane in n dimensions. The second plane intersects it (we hope) in a smaller set of "dimension $n-2$." Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

Column Vectors and Linear Combinations

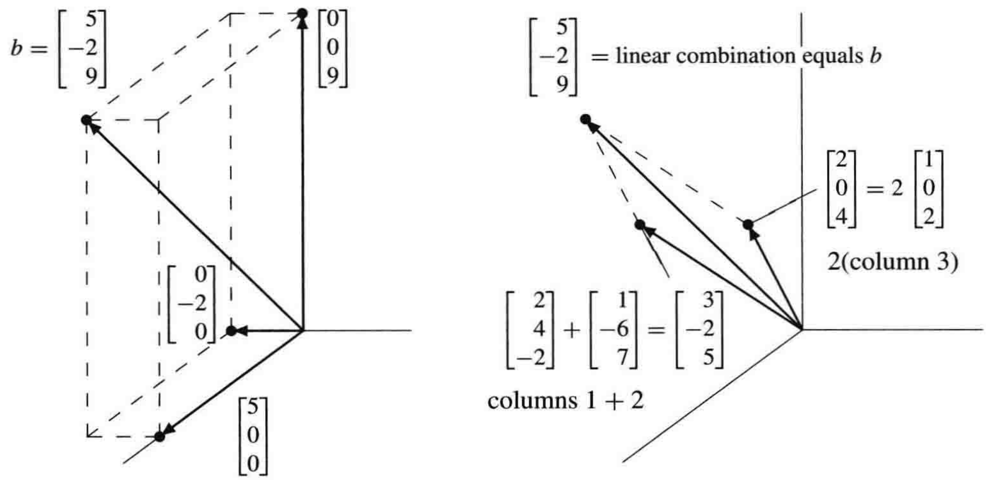
We turn to the columns. This time the vector equation (the same equation as (1)) is

$$\textbf{Column form} \qquad u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b. \tag{2}$$

Those are *three-dimensional column vectors*. ***The vector b is identified with the point whose coordinates are* 5, −2, 9**. Every point in three-dimensional space is matched to a vector, and vice versa. That was the idea of Descartes, who turned geometry into algebra by working with the coordinates of the point. We can write the vector in a column, or we can list its components as $b=(5,-2,9)$, or we can represent it geometrically by an arrow from the origin. You can choose *the arrow*, or *the point*, or *the three numbers*. In six dimensions it is probably easiest to choose the six numbers.

We use parentheses and commas when the components are listed horizontally, and square brackets (with no commas) when a column vector is printed vertically. What really matters is ***addition of vectors*** and ***multiplication by a scalar*** (a number). In Figure 1.4a you see a vector addition, component by component:

$$\textbf{Vector addition} \qquad \begin{bmatrix}5\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-2\\0\end{bmatrix}+\begin{bmatrix}0\\0\\9\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$



(a) **Add vectors along axes** (b) **Add columns $\mathbf{1+2+(3+3)}$**

Figure 1.4 The column picture: linear combination of columns equals b.

In the right-hand figure there is a multiplication by 2 (and if it had been -2 the vector would have gone in the reverse direction):

$$\textbf{Multiplication by scalars} \qquad 2\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}2\\0\\4\end{bmatrix}, \qquad -2\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}-2\\0\\-4\end{bmatrix}.$$

Also in the right-hand figure is one of the central ideas of linear algebra. It uses *both* of the basic operations; vectors are *multiplied by numbers and then added.* The result is called a ***linear combination***, and this combination solves our equation:

$$\textbf{Linear combination} \qquad \mathbf{1}\begin{bmatrix}2\\4\\-2\end{bmatrix} + \mathbf{1}\begin{bmatrix}1\\-6\\7\end{bmatrix} + \mathbf{2}\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}.$$

Equation (2) asked for multipliers u, v, w that produce the right side b. Those numbers are $u = \mathbf{1}$, $v = \mathbf{1}$, $w = \mathbf{2}$. They give the correct combination of the columns. They also gave the point $(1, 1, 2)$ in the row picture (where the three planes intersect).

Our true goal is to look beyond two or three dimensions into n dimensions. With n equations in n unknowns, there are n planes in the row picture. There are n vectors in the column picture, plus a vector b on the right side. The equations ask for a ***linear combination of the n columns that equals b***. For certain equations that will be impossible. Paradoxically, the way to understand the good case is to study the bad one. Therefore we look at the geometry exactly when it breaks down, in the **singular case**.

Row picture: Intersection of planes	***Column picture***: Combination of columns

The Singular Case

Suppose we are again in three dimensions, and the three planes in the row picture *do not intersect*. What can go wrong? One possibility is that two planes may be parallel. The equations $2u + v + w = 5$ and $4u + 2v + 2w = 11$ are inconsistent—and parallel planes give no solution (Figure 1.5a shows an end view). In two dimensions, parallel lines are the only possibility for breakdown. But three planes in three dimensions can be in trouble without being parallel.

The most common difficulty is shown in Figure 1.5b. From the end view the planes form a triangle. Every pair of planes intersects in a line, and those lines are parallel. The

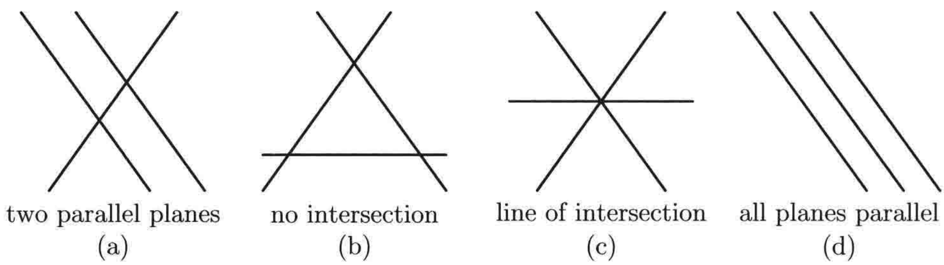


Figure 1.5 Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

third plane is not parallel to the other planes, but it is parallel to their line of intersection. This corresponds to a singular system with $b = (2, 5, 6)$:

$$\textbf{No solution, as in Figure 1.5b} \qquad \begin{aligned} u + v + w &= 2 \\ 2u \phantom{{}+v} + 3w &= 5 \\ 3u + v + 4w &= 6. \end{aligned} \tag{3}$$

The first two left sides add up to the third. On the right side that fails: $2 + 5 \neq 6$. Equation 1 plus equation 2 minus equation 3 is the impossible statement $0 = 1$. Thus the equations are ***inconsistent***, as Gaussian elimination will systematically discover.

Another singular system, close to this one, has an **infinity of solutions**. When the 6 in the last equation becomes 7, the three equations combine to give $0 = 0$. Now the third equation is the sum of the first two. In that case the three planes have a whole *line in common* (Figure 1.5c). Changing the right sides will move the planes in Figure 1.5b parallel to themselves, and for $b = (2, 5, 7)$ the figure is suddenly different. The lowest plane moved up to meet the others, and there is a line of solutions. Problem 1.5c is still singular, but now it suffers from ***too many solutions*** instead of too few.

The extreme case is three parallel planes. For most right sides there is no solution (Figure 1.5d). For special right sides (like $b = (0, 0, 0)$!) there is a whole plane of solutions—because the three parallel planes move over to become the same.

What happens to the ***column picture*** when the system is singular? It has to go wrong; the question is how. There are still three columns on the left side of the equations, and we try to combine them to produce b. Stay with equation (3):

Singular case: Column picture
Three columns in the same plane
Solvable only for b in that plane

$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b. \tag{4}$$

For $b = (2, 5, 7)$ this was possible; for $b = (2, 5, 6)$ it was not. The reason is that ***those three columns lie in a plane***. Then every combination is also in the plane (which goes through the origin). If the vector b is not in that plane, no solution is possible (Figure 1.6). That is by far the most likely event; a singular system generally has no solution. But

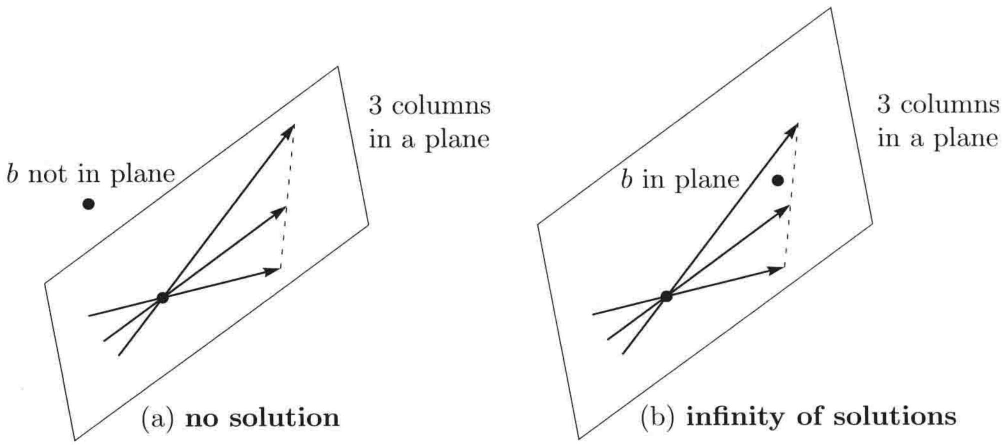


Figure 1.6 Singular cases: b outside or inside the plane with all three columns.