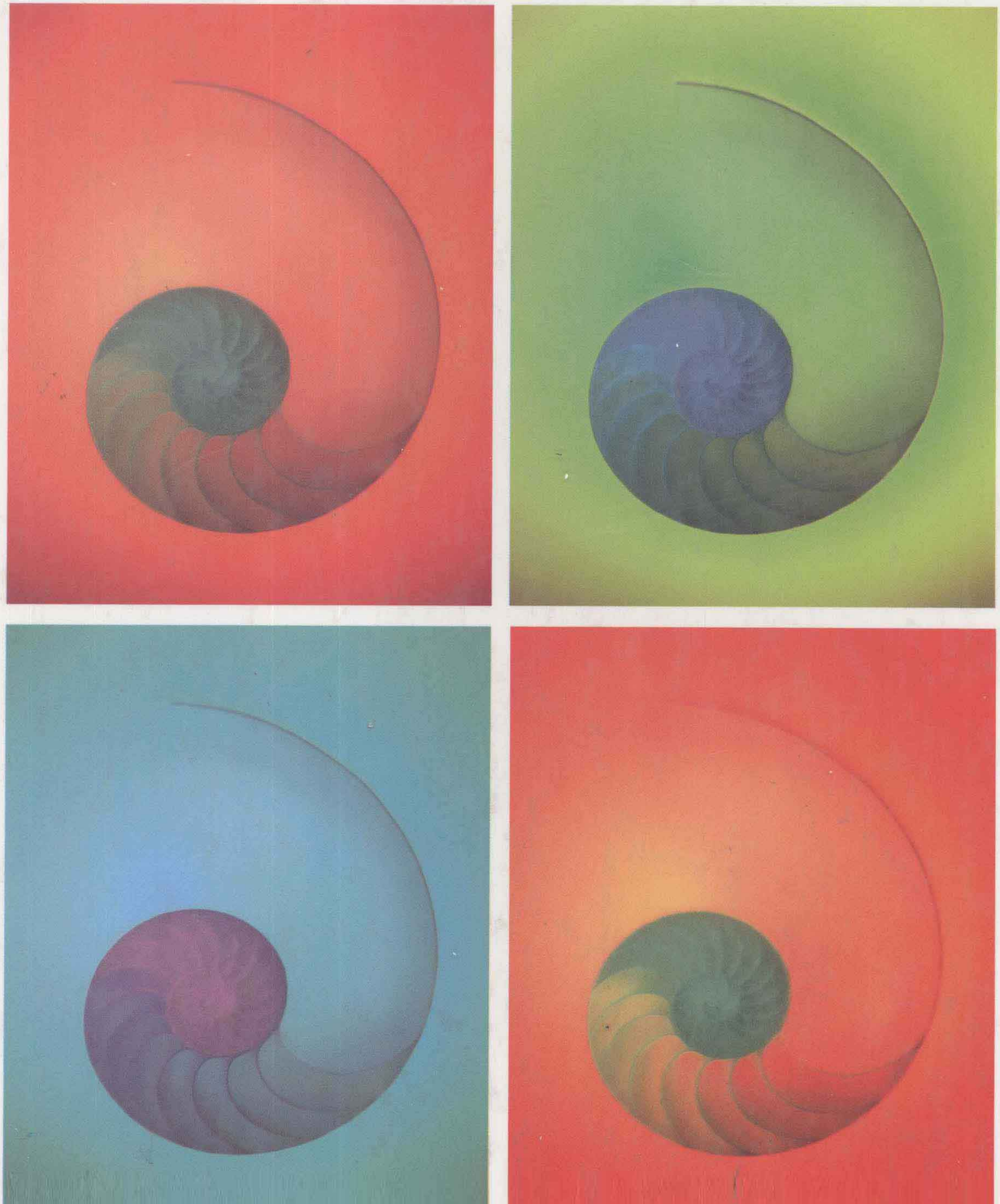
Karl J. Smith

The Nature of Mathema

Ninth Edition



The Nature of Mathematics

Ninth Edition

Karl J. Smith
Santa Rosa Junior College

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I dedicate this book with love to Melissa, Benjamin, and the light of my life, Hannah

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Preface

Like almost every subject of human interest, mathematics is as easy or as difficult as we choose to make it. Following this Preface, I have included a *Fable*, addressed directly to the student. I hope you take the time to read it, and then ponder why I call it a fable. Since the first edition, my goal has been, and continues to be, to create a positive attitude toward mathematics. But the world, the students, and the professors are very different today than they were when I began writing this book. This is a very different book from its first printing, and this edition is very different from the previous edition. The world of knowledge is more accessible today (via the World Wide Web) than at any time in history. Supplementary help is available on the World Wide Web, and can be accessed at the following Web address:

www.mathnature.com

All of the Web addresses mentioned in this book are linked to the above Web address. If you have access to a computer and the World Wide Web, check out this Web address. You will find links to several search engines, history, and reference topics, a keyword search, and a menu that allows you to choose a chapter and section. You will find, for each section, homework hints, a listing of essential ideas, projects, and links to related information on the Web.

This book was written for those students who need a mathematics course to satisfy the general university competency requirement in mathematics. Because of the university requirement, many students enrolling in a course that uses my book have postponed taking this course as long as possible. They dread the experience, and come to class with a great deal of anxiety. Rather than simply presenting the technical details needed to proceed to the next course, I have attempted to give insight into what mathematics is, what it accomplishes, and how it is pursued as a human enterprise. However, at the same time, I have included in this ninth edition a great deal of material to help students estimate, calculate, and solve problems *outside* the classroom or textbook setting.

I frequently encounter people who tell me about their unpleasant experiences with mathematics. I have a true sympathy for those people, and I recall one of my elementary school teachers who assigned additional arithmetic problems as punishment. This can only create negative attitudes toward mathematics, which is indeed unfortunate. If elementary school teachers and parents have positive attitudes toward mathematics, their children cannot help but see some of the beauty of the subject. I want students to come away from this course with the feeling that mathematics can be pleasant, useful, and practical—and enjoyed for its own sake.

The prerequisites for this course vary considerably, as do the backgrounds of students. Some schools have no prerequisites, whereas other schools have an intermediate algebra prerequisite. The students, as well, have heterogeneous backgrounds. Some have little or no mathematics skills; others have had a great deal of mathematics. Even though the usual prerequisite for using this book is intermediate algebra, a careful selection of topics and chapters would allow a class with a beginning algebra prerequisite to effectively study the material.

This book was written to meet the needs of all students and schools. How did I accomplish that goal? First, the chapters are almost independent of one another, and can be covered in any order appropriate to a particular audience. Second, the problems are designed to be the core of the course. There are problems that every student will find easy in order to provide the opportunity for success; there are also problems that are very challenging. Much interesting material appears in the problems, and students should get into the habit of reading (not necessarily working) all the problems whether or not they are assigned.

Level 1: mechanical or drill problems
Level 2: require understanding of the concepts
Level 3: require problem solving skills or original thinking

The major themes of this book are problem solving and estimation in the context of presenting the great ideas in the history of mathematics. I believe that *learning to solve problems is the principal reason for studying mathematics*. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in most textbooks is one form of problem solving, but students also should be faced with non-text-type problems. In the first section of this edition I introduce students to Pólya's problem-solving techniques, and these techniques are used throughout the book to solve non-text-type problems. Problem-solving examples (marked as PÓLYA'S METHOD examples) are found throughout the book. Also new to this edition are problems in *each* section that require Pólya's method for problem solving.

Students should learn the language and notation of mathematics. Most students who have trouble with mathematics do not realize that mathematics *does require hard work*. The usual pattern for most mathematics students is to open the book to the assigned page of problems, and begin working. Only after getting "stuck" is an attempt made to "find it in the book." The final resort is reading the text. In this book the students are asked not only to "do math problems," but also to "experience mathematics." This means it is necessary to become involved with the **concepts** being presented, not "just get answers." In fact, the advertising slogan "Mathematics Is Not a Spectator Sport" is an invitation which suggests that the only way to succeed in mathematics is to become involved with it. Students will learn to receive mathematical ideas through listening, reading, and visualizing. They are expected to present mathematical ideas by speaking, writing, drawing pictures and graphs, and demonstrating with concrete models. New to this edition is a category of problems in each section which is designated IN YOUR OWN WORDS, which provides practice in communication skills.

A Note for Instructors

Feel free to arrange the material in a different order from that presented in the text. I have written the chapters to be as independent of one another as possible. There is much more material than could be covered in a single course. This book can be used in classes designed for liberal arts, teacher training, finite mathematics, college algebra, or a combination of these.

I have written an extensive *Instructor's Manual* to accompany this book. It includes the complete solutions to all the problems (including the "Problem Solving" problems) as well as teaching suggestions and transparency masters. For those who wish to integrate the computer into the entire course, there are computer problems in both BASIC and LOGO to accompany each chapter.

Also available are sample tests, not only in hard copy form, but also in electronic form for both IBM and Macintosh formats.

Changes from the Previous Edition

It is difficult to think about changing a successful textbook. With this edition I have significantly updated the presentation. I have completely reworked the problem sets, and offer problem sets of uniform length (60 problems in each) to facilitate the assigning of problem sets. I have added over 100 pages of new material, including the following:

History of Mathematics
Binary Numeration System
Discrete Mathematics
Right-Triangle Trigonometry
Euler Circuits and Hamiltonian Cycles
Perimeter
Surface Area
Exponential Equations
Logarithmic Equations
Applications of Growth and Decay
Voting and Apportionment
Cumulative Frequencies
Conic Sections
Derivatives
Integrals
Mathematics in the Natural Sciences, Social Sciences, and Humanities

In addition, I have relocated the section on Problem Solving with Sets, and the chapter on The Nature of Sequences, Series, and Financial Management.

The principal change I have made with this edition is the integration of material in this book and material on the World Wide Web. Without computer access, the book completely stands alone, just as has been with the previous editions. But for those with access to the Web, there is a new world of enhancement accompanying this book. All of this material is found on our Web page at **www.mathnature.com.**

Acknowledgments

I appreciate the suggestions of the reviewers of this edition: Richard C. Andrews, Florida A&M University; Elaine Bouldin, Middle Tennessee State University; Frances J. Lane, Virginia State University; and Elaine I. Miller, University of Toledo.

One of the nicest things about writing a successful book is all of the letters and suggestions I've received. I would like to thank the following people who gave suggestions for previous editions of this book: Jeffery Allbritten, Brenda Allen, Nancy Angle, Peter R. Atwood, John August, Charles Baker, V. Sagar Bakhshi, Jerald T. Ball, Carol Bauer, George Berzsenyi, Daniel C. Biles, Jan Boal, Kolman Brand, Chris C. Braunschweiger, Barry Brenin, T. A. Bronikowski, Charles M. Bundrick, T. W. Buquoi, Eugene Callahan, Michael W. Carroll, Joseph M. Cavanaugh, James R. Choike, Mark

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Ann Ostberg did a superb job of checking all of the examples and checking the accuracy of the answers. I would especially like to thank Robert J. Wisner of New Mexico State for his countless suggestions and ideas over the many editions of this book; Tessa Avila, Craig Barth, Jeremy Hayhurst, Paula Heighton, Gary Ostedt, and Bob Pirtle of Brooks/Cole; as well as Jack Thornton, for the sterling leadership and inspiration he has been to me from the inception of this book to the present.

Finally, my thanks go to my wife, Linda, who has always been there for me. Without her this book would exist only in my dreams, and I would have never embarked as an author.

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To the Student: A Fable

Once upon a time, two young ladies, Shelley and Cindy, came to a town called Mathematics. People had warned them that this is a particularly confusing town. Many people who arrived in Mathematics were very enthusiastic, but could not find their way around, became frustrated, gave up, and left town.

Shelley was strongly determined to succeed. She was going to learn her way through the town. For example, in order to learn how to go from her dorm to class, she concentrated on memorizing this clearly essential information: she had to walk 325 steps south, then 253 steps west, then 129 steps in a diagonal (southwest), and finally 86 steps north. It was not easy to remember all of that, but fortunately she had a very good instructor who helped her to walk this same path 50 times. In order to stick to the strictly necessary information, she ignored much of the beauty along the route, such as the color of the adjacent buildings or the existence of trees, bushes, and nearby flowers. She always walked blindfolded. After repeated exercising, she succeeded in learning her way to class and also to the cafeteria. But she could not learn the way to the grocery store, the bus station, or a nice restaurant; there were just too many routes to memorize. It was so overwhelming! Finally she gave up and left town; Mathematics was too complicated for her.

Cindy, on the other hand, was of a much less serious nature. To the dismay of her instructor, she did not even intend to memorize the number of steps of her walks. Neither did she use the standard blindfold that students need for learning. She was always curious, looking at the different buildings, trees, bushes, and nearby flowers or anything else not necessarily related to her walk. Sometimes she walked down dead-end alleys in order to find out where they were leading, even if this was obviously superfluous. Curiously, Cindy succeeded in learning how to walk from one place to another. She even found it easy and enjoyed the scenery. She eventually built a building on a vacant lot in the city of Mathematics.*

* My thanks to Emilio Roxin of the University of Rhode Island for the idea for this fable.

History of Mathematics: An Overview

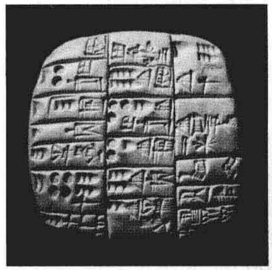
The study of mathematics can be organized as a history or story of the development of mathematical ideas, or it can be organized by topic. The intended audience of this book dictates that the development should be by topic, but mathematics involves real people with real stories, so you will find this text to be very historical in its presentation. This overview rearranges the material you will encounter in the text into a historical timeline. It is not intended to be read as a history of mathematics, but rather an overview to make you want to do further investigation. Sit back, relax, and use this overview as a starting place to expand your knowledge about the beginnings of some of the greatest ideas in the history of the world!

We have divided this history of mathematics into six chronological periods:

3000 BC to 601 BC	Egyptian, Babylonian, and Native American periods
600 BC to AD 499	Greek, Chinese, and Roman periods
500 to 1199	Hindu and Arabian period
1200 to 1599	Transition period
1600 to 1699	Century of Enlightenment
1700 to 1899	Early Modern period
1900 to the present	Modern period

Babylonian, Egyptian, and Native American Periods: 3000 BC to 601 BC

Mesopotamia is an ancient country located in southeast Asia between the lower Tigris and Euphrates Rivers. It is a part of modern Iraq. Mesopotamian mathematics refers to the mathematics of the ancient Babylonians, and this mathematics is sometimes referred to as Sumerian mathematics. Over 50,000 tablets from Mesopotamia have been found, and are exhibited at major museums around the world. Interesting readings about Babylon can be found by reading a history of mathematics book, such as *An Introduction to the History of Mathematics,* 6th edition, by Howard Eves (New York: Saunders, 1990) or by looking at the many sources on the World Wide Web. You can find links to these Web sites, as well as all the Web sites mentioned in this book, by looking at the Web page for this text:



Sumerian clay tablet

WWW **www.mathnature.com** History

Babylonian, Egyptian, and Native American Periods: 3000 BC to 601 BC

Cultural Events

3000	First Dynasty of the Ancient Kingdom of Egypt
2800	The Great Pyramid
2580	Cheops' Pyramid
2500	Isis and Osiris cult in Egypt
1900	Epic of Gilgamesh
1700	Stonehenge
1500	First alphabets created
1495	Obelisk of Thothmes at Karnak
1300	Approximate beginning of Iron Age
1250	Moses leads exodus from Egypt
1200	Trojan War
1000	Phoenicians invent alphabet
850	Homer: *Iliad* and *Odyssey*
753	Rome founded

Mathematical Events

3000	Chinese arithmetic and astronomy under Huang-ti
2900	Egyptian scribes use a hieratic script
2850	Egyptian simple grouping system
2200	Mathematical tablets of Nippur: oldest example of a magic square
1850	Moscow papyrus: 25 mathematical problems
1700	Early Babylonian tablet (Plimpton 322)
1650	Egyptian Astrological signs
1350	Rollins papyrus: elaborate mathematical problem
1105	The *Chou-pei,* major Chinese text on mathematics

This Web page can be your access to a world of information by using the links provided there. Now might be a good time to visit this Web page.

The mathematics of this period was very practical and it was used in construction, surveying, record keeping, and in the creation of calendars. The culture of the Babylonians reached its height about 2500 BC, and in about 1700 BC, King Hammurabi formulated a famous code of law. In 330 BC, Alexander the Great conquered Mesopotamia, ending the great Babylonian empire. Even though there was a great deal of political and social upheaval during this period, there was a continuity in the development of mathematics from ancient time to the time of Alexander.

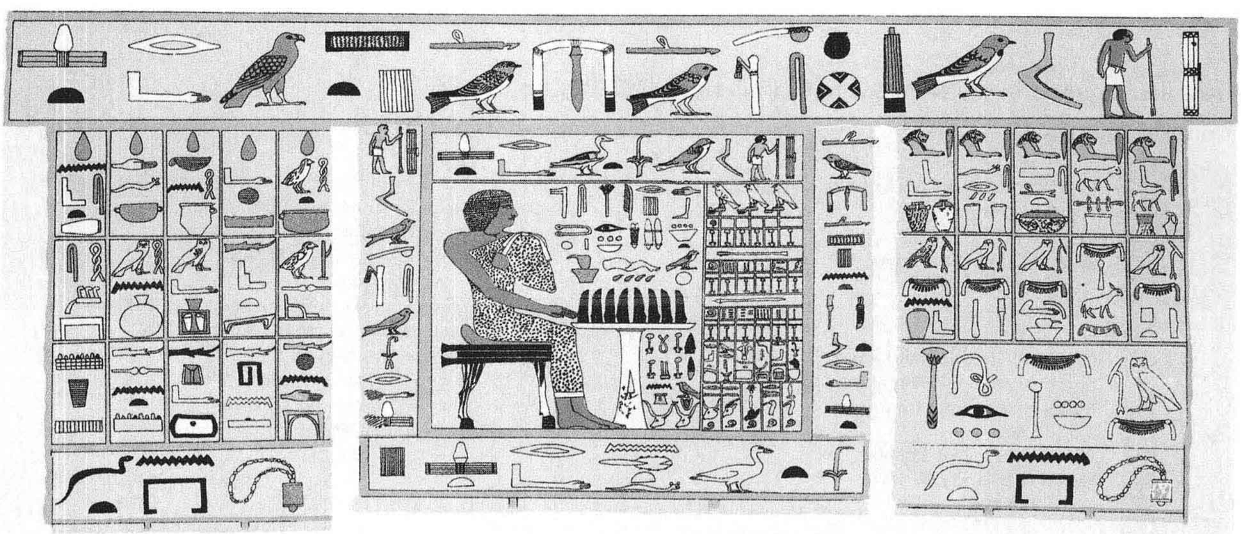
The main information we have about the civilization and mathematics of the Babylonians is their numeration system, which we introduce in the text in Section 3.1 (p. 120). The Babylonian numeration system was positional with base 60. They did not have a 0 symbol, but they did represent fractions, squares, square roots, cubes, and cube roots. We have evidence that they knew the quadratic formula (Chapter 5, p. 267), and they had stated algebraic problems verbally. The base 60 system of the Babylonians led to a division of a circle into 360 equal parts that today we call degrees, and each degree was in turn divided into 60 parts that today we call seconds. The Greek astronomer Ptolemy (AD 85 − 165) used this system from the Babylonians, which is no doubt why we have minutes, seconds, and degree measurement today.

The Egyptian civilization existed from about 4000 BC, and was less influenced by foreign powers than was the Babylonian civilization. Egypt was divided into two kingdoms until about 3000 BC, when the ruler Menes unified Egypt and consequently became known as the founder of the first dynasty in 2500 BC. This was the pyramid building period, and the Great Pyramid of Cheops was built around 2600 BC (Chapter 6, pp. 364 and 368–369; see The Riddle of the Pyramids p. 370).

The Egyptians developed their own pictorial way of writing, called *hieroglyphics,* and their numeration system was consequently very pictorial (Chapter 3, p. 116).

The Egyptian numeration system is an example of a simple grouping system. Although they were able to write fractions, they used only unit fractions (p. 117). Like the Babylonians, they had not developed a symbol for zero (p. 121). Since the writing of the Egyptians was on papyrus, and not on tablets as with the Babylonians, most of the written history has been lost. Our information comes from the Rhind papyrus (p. 118), discovered in 1858 and dated to about 1700 BC, and the Moscow papyrus (p. 118), which has been dated to about the same time period.

The mathematics of the Egyptians remained remarkably unchanged from the time of the first dynasty to the time of Alexander the Great, who conquered them in 332 BC. The Egyptians did surveying using a unique method of stretching rope, so they referred to their surveyors as "rope stretchers" (pp. 367–368). The basic unit used by the



Egyptian hieroglyphics: Inscription and relief from the grave of Prince Rahdep (ca. 2800 B.C.)

Egyptians for measuring length was the *cubit,* which was the distance from a person's elbow to the end of the middle finger. A *khet* was defined to equal 100 cubits; khets were used by the Egyptians when land was surveyed. The Egyptians did not have the concept of a variable, and all their problems were verbal or arithmetic. Even though they solved many equations, they used the word *AHA* or *heap* in place of the variable (p. 278). For an example of an Egyptian problem, see Ahmes' dilemma (p. 17) and the statement of the problem in terms of Thoth, an ancient Egyptian god of wisdom and learning.

The Egyptians had formulas for the area of a circle (p. 437) and the volume of a cube, box, cylinder, and other figures. Particularly remarkable is their formula for the volume of a truncated pyramid of a square base, which in modern notation is:

$$V = \tfrac{h}{3}(a^2 + ab + b^2)$$

where h is the height and a and b are the sides of the top and bottom. Even though we are not certain the Egyptians knew of the Pythagorean theorem (p. 201), we believe they did because the rope stretchers had knots on their ropes that would form right triangles. They had a very good reckoning of the calendar, and knew that a solar year was approximately $365\frac{1}{4}$ days long. They chose as the first day of their year, the day on which the Nile would flood.

The next chapter in the history of mathematics is from the Babylonian and Egyptian civilizations to the Greeks because of the conquests of Alexander the Great. Contemporaneous with the great civilizations in Mesopotamia was the great Mayan civilization in what is now Mexico. A Mayan timeline is shown in Table 1.

Table 1 Mayan Timeline

1200–1000 BC	Olmec
1800–900 BC	Early Preclassic Maya
900–300 BC	Middle Preclassic Maya
300 BC– AD 250	Late Preclassic Maya
250–600	Early Classic Maya
699–900	Late Classic Maya
900–1500	Post Classic Maya
1500–1800	Colonial period
1821–present	Mexico

Just as with the Mesopotamian civilizations, the Olmeca and Mayan civilizations lie between two great rivers, in this case the Grijalva and the Papaloapa. Sometimes the Olmecas are referred to as the Tenocelome. The Olmeca culture is considered the mother culture of the Americas. What we know about the Olmecas centers around their art. We do know they were a farming community. The Maya civilization began around 2600 BC and gave rise to the Olmecs. A written hieroglyphic language had been developed by 700 BC and they had a very accurate solar calendar. The Mayan culture had developed a positional numeration system (p. 122).

You will find influences from *this* period discussed throughout the book. Look in the following places:

Babylonian

Egyptian

Others

Mayans used zero as placeholder, Chapter 4, p. 210
Beginnings of algebra and algebraic ideas (al-Khwârizmî), Chapter 5, p. 238
Quadratic equation, Section 5.4, Problems 59–60, Historical Questions, pp. 271–272
Louvre tablet, Section 12.1 Historical Question/Problem 59, p. 812

Greek, Roman, and Chinese Periods: 600 BC to 499 AD

Greek mathematics began in 585 BC when Thales, the first of the Seven Sages of Greece (625 – 547 BC) traveled to Egypt.* The Greek civilization was most influential in our history of mathematics. So striking was their influence that the historian Morris Kline declares, "One of the great problems of the history of civilization is how to account for the brilliance and creativity of the ancient Greeks."† The Greeks settled in Asia Minor, modern Greece, southern Italy, Sicily, Crete, and North Africa. They replaced the various hieroglyphic systems with the Phoenician alphabet and with that were able to become more literate and more capable of recording history and ideas. The Greeks had their own numeration system (p. 122). They had fractions and some irrational numbers, especially π.

The great mathematical contributions of the Greeks are Euclid's *Elements* (p. 329) and Apollonius' *Conic Sections* (p. 742). Greek knowledge developed in several centers or schools. (See Figure 1 for a depiction of one of these centers of learning.) The first was founded by Thales (ca. 640 – 546 BC) and known as the Ionian in Miletus. It is re-

* The Seven Sages in Greek history refer to Thales of Miletus, Bias of Priene, Chilo of Sparta, Cleobulus of Rhodes, Periander of Corinth, Pittacus of Mitylene, and Solon of Athens; they were famous because of their practical knowledge about the world and how things work.

† p. 24, *Mathematical Thought from Ancient to Modern Times* by Morris Kline (New York: Oxford University Press, 1972).

Greek and Roman Periods: 600 BC to 499 AD

Cultural Events

538	Persians capture Babylon
500	Pindar's *Odes*
480	Siddhartha, the Buddha, delivers his sermons in Deer Park
323	Alexander the Great completes his conquest of the known world
218	Hannibal crosses the Alps
200	Rosetta Stone engraved
100	Birth of Julius Caesar
20	Virgil: *Aeneid*
4	Birth of Christ
▲ BC-AD ▼	
200	Goths invade Asia Minor
324	Founding of Constantinople
400	Augustine, *Confessions*
476	Fall of Rome

Mathematical Events

585	Thales, founder of Greek geometry
540	The teachings of Pythagoras
500	Sulvasutras: Pythagorean numbers
450	Zeno: paradoxes of motion
425	Theodorus of Cyrene: irrational numbers
384	Aristotle: logic
380	Plato's Academy: logic
323	Euclid: geometry, perfect numbers
300	First use of Hindu numeration system
230	Sieve of Eratosthenes
225	Archimedes: circle, pi, curves, series
180	Hypsicles: number theory
60	Geminus: parallel postulate
▲ BC-AD ▼	
50	Negative numbers used in China
75	Heron: measurements, roots, surveying
100	Nichomachus: number theory
150	Ptolemy: trigonometry
200	Mayan calendar
250	Diophantus: number theory, algebra
300	Pappus: *Mathematical Collection*
410	Hypatia of Alexandria: first woman mentioned in the history of mathematics
480	Tsu Ch'ung-chi approximates π as 355/113

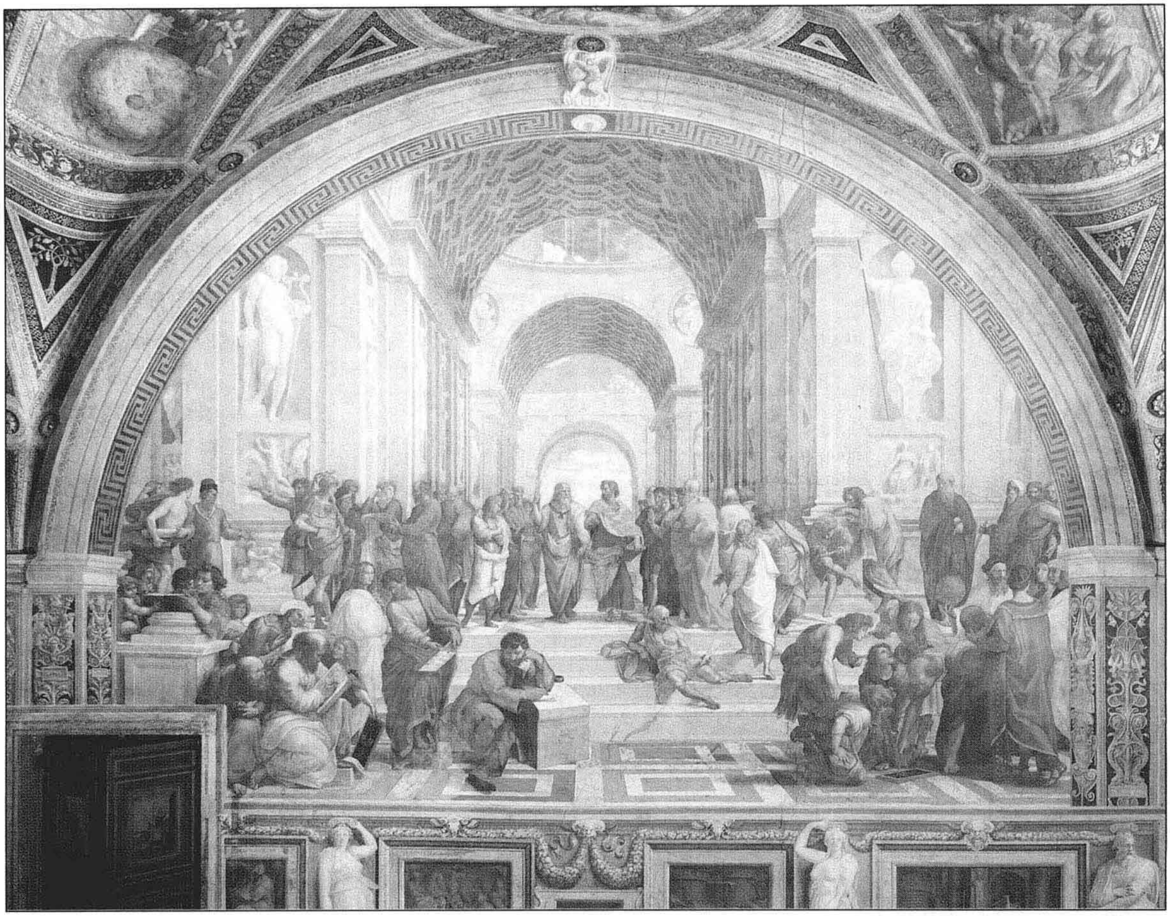


Figure 1 **The school of Athens by Raphael, 1509. This fresco includes portraits of Raphael's contemporaries, and demonstrates the use of perspective. Note the figures in the lower right, who are, no doubt, discussing mathematics.**

ported that while he was traveling and studying in Egypt, he calculated the heights of the pyramids by using similar triangles, just as we do in Example 5 of Section 6.5 (p. 364). You can read about these two great Greek mathematicians in *Mathematics Thought from Ancient to Modern Times* by Morris Kline. You can also refer to the World Wide Web at **www.mathnature.com**.

Between 585 BC and 352 BC, schools flourished establishing the foundations for the way knowledge is organized today. Figure 2 shows each of the seven major schools, along with each school's most notable contribution. Links to textual discussion are shown within each school of thought, along with the principal person associated with each of these schools. Books have been written about the importance of each of these Greek schools, and several links can be found at **www.mathnature.com**.

One of the three greatest mathematicians in the entire history of mathematics was Archimedes (287−212 BC). His accomplishments are truly remarkable, and you should seek out other sources about his life. He invented a pump (the Archimedian screw), military engines and weapons, and catapults; in addition, he used a paraboloidal mirror as a weapon by concentrating the sun's rays on invading Roman ships. "The most famous of the stories about Archimedes is his discovery of the method of testing the debasement of a crown of gold. The king of Syracuse had ordered the crown. When it was delivered, he suspected that it was filled with baser metals and sent it to Archimedes to

Greek Schools (585–352 B.C.)

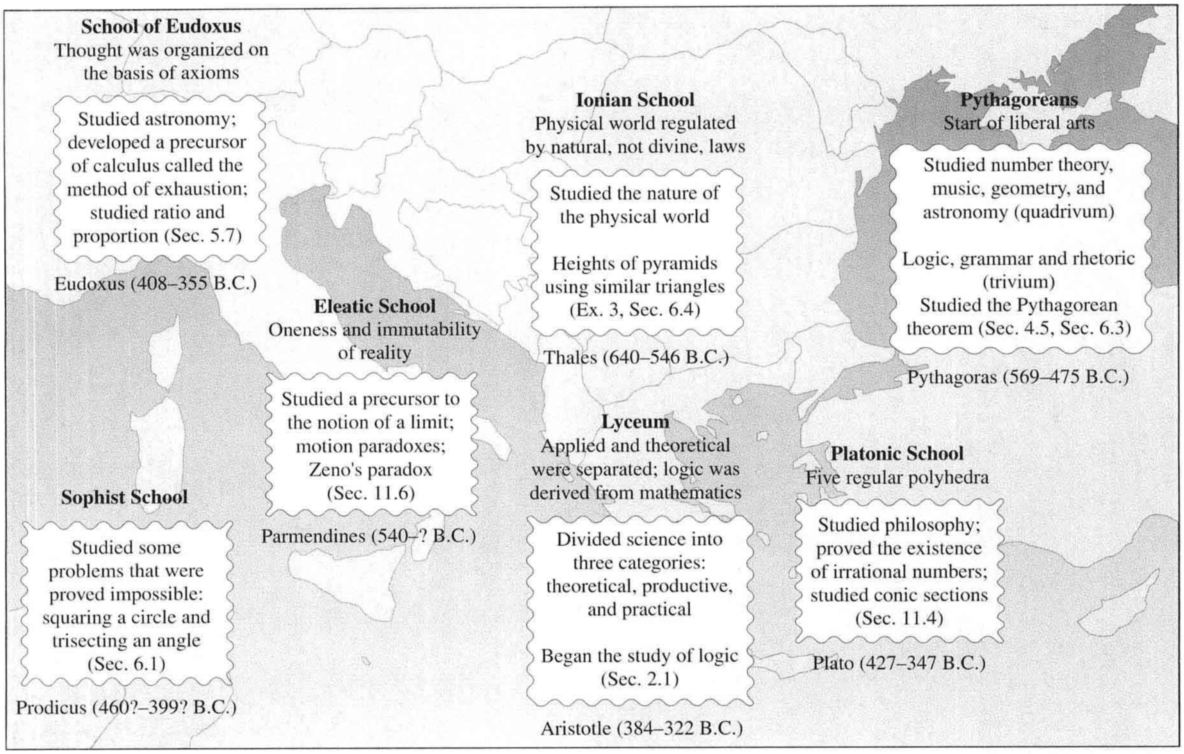


Figure 2 **Greek schools from 585 B.C. to 352 B.C.**

devise some method of testing the contents without, of course, destroying the workmanship. Archimedes pondered the problem; one day while bathing he observed that his body was partly buoyed up by the water and suddenly grasped the principle that enabled him to solve the problem. He was so excited by this discovery that he ran out into the street naked shouting 'Eureka!' ('I have found it!'). He had discovered that a body immersed in water is buoyed up by a force equal to the weight of the water displaced, and by means of this principle was able to determine the contents of the crown."*

The Romans conquered the world, but their mathematical contributions were minor. We introduced the Roman numerals in Section 3.1 (p. 118), and their fractions were based on a duodecimal (base 12) system. The unit of weight was the *as* and 1/12 of this was the *uncia*, from which we get our measurements of *ounce* and *inch*, respectively. They improved on our calendar, and set up the notion of a leap year every four years. The Julian calendar was adopted in 45 BC. The Romans conquered Greece and Mesopotamia and in 47 BC, they set fire to the Egyptian fleet in the harbor of Alexandria. The fire spread to the city and burned the library, destroying two and a half centuries of book collecting, including all the important knowledge of the time.

Another great world civilization existed in China where they also developed a decimal numeration system and used a decimal system with symbols 1, 2, 3, … , 9, 10, 100, 1000, and 10000. Calculations were performed using small bamboo counting rods, which eventually evolved into the abacus (p. 128). Our first historical reference to the Chinese culture is the yin-yang symbol (p. 65), which has its roots in ancient cosmology. The original meaning is representative of the mountains, both the bright and the dark side. The "yin" represents the female, or shaded, aspect:

* *Ibid*, pp. 105–106.

the earth, the darkness, the moon, passivity. The "yang" represents the male, light, sun, heaven, and the active principle in nature. These words can be traced back to the Shang and Chou Dynasty (1550 − 1050 BC), but most scholars credit them to the Han Dynasty (220 − 206 BC). One of the first examples of a magic square comes from the Lo River around 200 BC, where legend tells us that the emperor Yu of the Shang Dynasty received a magic square on the back of a tortoise's shell (see p. 59).

From 100 BC to 100 AD the Chinese described the motion of the planets, as well as what is the earliest known proof of the Pythagorean theorem (p. 201). The longest surviving and most influential Chinese math book is dated from the beginning of the Han Dynasty around AD 50. It includes measurement and area problems, proportions, volumes, and some approximations for π. Sun Zi (ca. AD 250) wrote his mathematical manual, which included the "Chinese remainder problem": Find n so that upon division by 3, you obtain a remainder of 2; upon division by 5 you obtain a remainder of 3, and upon division by 7 you obtain a remainder of 2. His solution: Add 140, 63, and 30 to obtain 233, and subtract 210 to obtain 23 (see p. 226). Zhang Qiujian (ca. AD 450) wrote a mathematics manual that included a formula for summing the terms of an arithmetic sequence, along with the solution to a system of two linear equations in three unknowns. The problem is the "One Hundred Fowl Problem," included in Problem Set 4.7 (p. 226). At the end of this historic period, the mathematician and astronomer Wang Xiaotong (ca. AD 626) solved cubic equations by generalization of an algorithm for finding the cube root.

Check **www.mathnature.com** for links to many excellent sites on Greek mathematics. In this text, we have included the following stories and references:

Greeks

Aristotle (384 BC), logic began, Chapter 2, p. 64
Greek numeration system, Historical Question, p. 122
Pythagoreans as a secret society, Chapter 4, pp. 195–196
Hypatia (370 − 415), Chapter 5, p. 284
Diophantus' age, Section 5.8 Historical Question, p. 308
Euclid's *Elements* written/foundations for geometry set, Chapter 6, p. 329
Apollonius (ca. 262 − 190 BC), Chapter 11, p. 742
Zeno's paradox, Chapter 11, p. 766
Archimedes (ca. 300 B.C.), area of circle, p. 768

Romans

Roman numeration system, Chapter 3, p. 118

Chinese

Shang Dynasty/Emperor Yu/Lo-shu magic square, Chapter 1, p. 59
Yin-yang symbol/duality of nature, Chapter 2, p. 65
Pascal's triangle/Chinese manuscript, Chapter 3 Historical Question, p. 124
Abacus, Section 3.2, Problems 44–57, pp. 128–129
Chou Pei discovery of the Pythagorean theorem, Chapter 4, p. 201
One Hundred Fowl problem/Chinese puzzle, Chapter 4, Problem 58, p. 226
Chinese remainder problem, Chapter 4, Problem 59, p. 226

Hindu and Arabian Period AD 500 to 1199

Much of the mathematics that we read in contemporary mathematics textbooks ignores the rich history of this period. Included on the World Wide Web are some very good sources; check our Web site, **www.mathnature.com** for some links. The Hindu civilization dates back to 2000 BC, but the first recorded mathematics was during the Sulvasutra period from 800 BC to AD 200. In the third century, Brahmi symbols were used for 1, 2, 3, . . . , 9 and are significant because there was a single symbol for each number. There was no zero or positional notation at this time, but by AD 600 the Hindus used the Brahmi symbols with positional notation. We will discuss a numeration system that eventually evolved from these Brahmi symbols in Section 3.2 (Chapter 3, p. 125). For fractions, the Hindus used sexagesimal positional notation in astronomy, but in other applications they used a ratio of integers and wrote $\frac{3}{4}$ (without the fractional bar we use today). The first mathematically important period was the second period from AD 200 − 1200. The important mathematicians of this period are Āryabhata (AD 476 − 550), Brahmagupta (AD 598 − 670), Māhavira, (9th century), and Bhaskara (1114 − 1185). We include some historical questions from Bhaskara and Brahmagupta in Chapter 5 (p. 288).

The Hindus developed arithmetic independently of geometry and had a fairly good knowledge of rudimentary algebra. They knew that quadratic equations had two solutions, and had a good approximation for π. Astronomy motivated their study of trigonometry. Around 1200, scientific activity in India declined, and mathematical progress

Hindu and Arabian American periods AD 500 to 1199

Cultural Events

500	First plans of the Vatican Palace in Rome
610	Mohammed's vision
697	Northern Irish submit to Roman Catholicism
800	Charlemagne crowned emperor of Holy Roman Empire
832	Utrecht Psalter
	Beginning of Carolingian dynasties
870	First printed book
871	Alfred the Great
	Schism of the Church
900	Vikings discover Greenland
	Reign of Otto I
950	Beginning of the Dark Ages
990	Development of systematic musical notation
993	First canonization of saints
980-1002	Emperor Otto II
1003	Leif Erickson crosses Atlantic to Vinland
1008	World's first novel, *Tale of Genji*
1028	School of Chartres
1050	Normans penetrate England
1054	Macbeth defeated at Dunsinane
1065	Consecration of Westminster Abbey
1086	Chinese use moveable type to print books
1088	First modern university
1096	Start of the First Crusade
1110	Chinese invent the playing card
1125	Commencement of troubadour music
1154	Beginning of Plantagenet reign
1165	Maimonides: *Mishneh Torah*
1186	Domesday Book, tax census ordered by William the Conqueror

Mathematical Events

630	Brahmagupta: algebra, astronomy
710	Bede: calendar, finger arithmetic
750	First use of zero symbol
810	Mohammed ibn Mûsâ al-Khwârizmî coins term 'algebra,'
	Hindu numerals
850	Mahavira: arithmetic, algebra
870	Iâbit ibn Qorra: algebra, magic squares, amicable numbers
900	Abû Kâmil: Algebra, Bakhshali manuscript
976	Oldest example of written numerals in Europe
980	Abu'l-wefa: constructions, trig tables
999	Pope Sylvester II (Gerbert): arithmetic, pi approximated as $\sqrt{8} \approx 2.83$
1000	Sridhara recognizes the importance of the zero
1020	Al-Karkhî: algebra
1075	Game of rithmomachia
1110	Omar Khayyám: cubic equations, Pascal's triangle
1120	Bhāskara
1125	Earliest account of a mariner's compass
1150	Bhāskara: algebra
1175	Averroës: trigonometry, astronomy

ceased and did not revive until the British conquered India in the 18th century.

The Arabs invited Hindu scientists to settle in Baghdad, and when Plato's Academy closed in AD 529, many scholars traveled to Persia and became part of the Arab world. Omar Khayyám (1048 − 1122) and Nasîr-Eddin (1201 − 1274) worked freely with irrationals, which contrasted with the Greek idea of numbers. What we call Pascal's triangle dates back to this period (see Figure 1.4, p. 9). The word "algebra" comes from the Arabs in a book by the astronomer Mohammed ibn Musa al-Khwârizmî (780 − 850) entitled *Al-jabr w'al muqâbala* (see p. 238). Al-Khwârizmî solves quadratic equations and knows there are two roots, and even though they gave algebraic solutions of quadratic equations, they explained their work geometrically. The Arabs solved some cubics, but could solve only simple trigonometric problems. As stated by Morris Kline, "The Arabs made no significant advance in mathematics. What they did was absorb Greek and Hindu mathematics, preserve it, and ultimately . . . transmit it to Europe." *

Check **www.mathnature.com** for links to many related sites. In this text, we have included the following stories and references:

Hindus

Numeration system, Chapter 3, pp. 125–129

Bhaskara, Chapter 5 Historical Questions and Problems 56 − 58, p. 288

Brahmagupta, Chapter 5 Historical Questions and Problems 59 − 60, p. 588

* *Ibid*, pp. 197.

Code of Manu (AD 100) Chapter 9, p. 588. This is an ethical law of classical Hinduism. It teaches that the caste system is divinely ordained. It also teaches the various stages through which a man is expected to pass in a successful life: student, householder, hermit, and wandering beggar. These states are only for twice-born men. Women should stay in the home under the protection and control of the chief male in the household. The code requires the cultivation of pleasantness, patience, control of mind, non-stealing, purity, control of senses, intelligence, knowledge, truthfulness, and nonirritability. The killing of cows is listed among the greatest of sins.

Arabs

Omar Khayyám/Pascal's triangle, Section 1.1, Example 7, p. 13

Mohammed ibn Musa al-Khwârizmî and the origin of the word "algebra," Chapter 3, p. 238

Transition Period: 1200 to 1599

Mathematics during the Middle Ages was transitional between the great early civilizations and the Renaissance. In the 1400s, the Black Death killed over 70% of the European population. The Turks conquered Constantinople, and many Eastern scholars traveled to Europe, spreading Greek knowledge as they traveled. The period from 1400 to 1600, known as the Renaissance, forever changed the intellectual outlook in Europe, and raised mathematical thinking to a new level. Johann Gutenberg's invention of printing with moveable type in 1450 changed the complexion of the world. Linen and cotton paper, which the Europeans learned about from the Chinese through the Arabians, came at precisely the right historical moment. The first printed edition of Euclid's *Elements* in a Latin translation appeared in 1482. Other early printed books were Apollonius' *Conic Sections*, Pappus' works, and Diophantus' *Arithmetica*.

The first breakthroughs in mathematics were by artists who discovered mathematical perspective (see pp. 401–402 and 891–892). The theoretical genius in mathematical perspective was Leone Alberti (1404−1472). He was a secretary in the Papal Chancery writing biographies of the saints, but his work on the laws of perspective, *Della Pictura*, was a masterpiece. He said, "Nothing pleases me so much as mathematical investigations and demonstrations, especially when I can turn them into some useful practice

Transition period: 1200 to 1449

Cultural Events

1206	Genghis Khan becomes chief prince of the Mongols
1209	Francis of Assisi initiates brotherhood
1233	Start of the Papal inquisition
1240	Amiens Cathedral rebuilt
1273	Thomas Aquinas: *Summa Theologicae*
1275	Moses de Leon: *Zohar*, major source for the cabala
1299	Florentine bankers are forbidden to use Hindu numerals
1307-21	Dante: *Divine Comedy*
1321	Chaucer
1322	The pope forbids the use of counterpoint in church music
1347-51	Approximately 75 million die of the Black Death
1364	Aztecs build Tenochtitlán
1378	Beginning of the Great Schism
1390	Chaucer: *Canterbury Tales*
1396	Metal type used for printing
1417	End of Great Schism
1420	Gutenberg and Kostner invent printer with moveable type
1429	Joan of Arc raises siege of Orleans
1435	Rogier Van der Weyden
1436	Fra Angelico begins frescoes at San Marco

Mathematical Events

1202	Fibonacci: arithmetic, algebra, geometry, sequences, *Liber Abaci*
1250	Sacrobosco: Hindu-Arabic numerals
1260	Campanus translates Euclid
1267	Roger Bacon: *Opus*
1280	Geometry used as the basis of painting
1281	Li Yeh introduces notation for negative numbers
1303	Chu Shi-Kie: algebra, solutions of equations, Pascal's triangle
1325	Thomas Bradwardine: arithmetic, geometry, star polygons
1360	Nicole Oresme: coordinates, fractional exponents
1400	In Florence, commercial activity results in several books on mercantile arithmetic
1425	Use of perspective gives depth to Renaissance painting
1435	Ulugh Beg: trig tables