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JOÃO B. PROLLA
IMECC, Universidade Estadual de Campinas, Brazil

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APPROXIMATION OF VECTOR VALUED FUNCTIONS

PREFACE

This work deals with the many variations of the Stone-Weierstrass Theorem for vector-valued functions and some of its applications. For a more detailed description of its contents see the Introduction and the Table of Contents. The book is largely self-contained. The amount of Functional Analysis required is minimal, except for Chapter 8. But the results of this Chapter are not used elsewhere. The book can be used by graduate students who have taken the usual first-year real and complex analysis courses.

The treatment of the subject has not appeared in book form previously. Even the proof of the Stone-Weierstrass Theorem is new, and due to S. Machado. We also give results in nonarchimedean approximation theory that are new and extend the Dieudonné-Kaplansky Theorem to nonarchimedean vector-valued function spaces.

I thank Professor Silvio Machado, from the Universidade Federal do Rio de Janeiro, for his valuable comments and remarks on the subject. Without his help this would be a different and poorer book. I thank also Professor Leopoldo Nachbin, from the Universidade Federal do Rio de Janeiro and the University of Rochester, whose advice and encouragement was never failing.

Finally, I wish to thank Angelica Marquez and Elda Mortari for typing this monograph.

JOÃO B. PROLLA

Campinas, April 1977

INTRODUCTION

The typical problem considered in this book is the following. One is given a vector subspace W of a locally convex space L of continuous vector-valued functions, which is a module over an algebra A of continuous scalar-valued functions, and the problem is to describe the closure of W in the space L.

In chapter 1 we start with the case in which $L=C(X;E)$ with the compact-open topology. When the algebra A is self-adjoint, the solution of the above problem is given by the Stone-Weierstrass theorem for modules. A very elegant and direct proof due to S. Machado (see [38]) is presented here. As a corollary one gets the classical Stone-Weierstrass theorem for self-adjoint subalgebras of $C(X;\mathbb{C})$. When the algebra A is not self-adjoint, a solution of the problem is given by Bishop's theorem. The proof that we include here is again due to S. Machado (see [37]). The main idea is to use a "strong" Stone-Weierstrass theorem for the real case plus a transfinite argument. This is done in Machado's paper via Zorn's Lemma. Here we use the transfinite induction process found in the original paper of Bishop (see [8]). We prefer this new method over de Brange's technique, because it can be applied to other situations in weighted approximation theory, namely where measure theoretic tools are either painful to apply or not available at all. In § 9 of this Chapter we treat a special case of vector fibrations, and prove in this context a "strong" Stone-Weierstrass theorem due to Cunningham and Roy (see [15]). This result is used in the next section to characterize extreme functionals. As corollaries, we get the Arens-Kelley theorem for scalar-valued functions, and Singer's theorem (vector-valued case). The results of Buck [12] and Ströbele [63] are also obtained. In an appendix we treat the non locally convex case.

Chapter 2 deals with vector-valued versions of Dieudonné's theorem on the approximation of functions of two variables by means of finite sums of products of functions of one variable (see [18]).

Chapter 3 is devoted to Tietze type extension theorems for vector-valued functions defined on compact subsets of a completely regular Hausdorff space. A typical result says that, if $Y \subset X$ is a compact subset of a completely regular space X, and E is a Fréchet space, then $C_b(X;E)|Y = C(Y;E)$.

The subject matter of chapter 4 is the notion of polynomial algebras. This notion was introduced in Pełczýnski [47], and the name is due to Wulbert (cf. Prenter [49]). In his definition Pełczyńsky used multilinear mappings, whereas Wulbert used polynomials. A third equivalent definition is given in Blatter [4]. We present here Stone-Weierstrass theorem for polynomial algebras. As a corollary we get the infinite dimensional version of the Weierstrass polynomial approximation theorem. Pełczyński attributes this result to S. Mazur (unpublished) in the case of Banach spaces. A much strengthened form of Mazur's result was proved in the joint paper Nachbin, Machado, Prolla [46], namely that the polynomials of finite type from a real locally convex space into another are dense in the space of all continuous function with the compact-open topology. Prenter [48] established Mazur's result for separable

Hilbert spaces. In this chapter we also prove Bishop's theorem for polynomial algebras using the definition given by Pełczyński. It remains an open problem for the more general polynomial algebras. Chapter 4 ends with a study of the approximation of compact linear operators by polynomials of finite type.

In Chapter 5 we are concerned with weighted approximation of vector-valued functions, i.e., with the Bernstein-Nachbin approximation problem. We extend the fundamental work of Nachbin (see for example [43]) from the real or self-adjoint complex case to the general complex case, in the same way that Bishop's theorem generalizes the Stone-Weierstrass theorem. In the joint paper with S. Machado [40], we accomplished this for vector fibrations. Here, however, we restrict ourselves to the particular case of vector-valued functions. As a corollary to our solution of the Bernstein-Nachbin approximation problem we get a strengthened version of Kleinstück's solution of the bounded case (see [35]) of Bernstein-Nachbin problem, as well as of Bishop's theorem for weighted spaces proved by Prolla [51]. The result of Summers [64] for scalar-valued functions is likewise generalized.

In the final two paragraphs of Chapter 5 we study the problem of completeness of Nachbin spaces and the characterization of the dual space of a Nachbin space.

In an appendix to Chapter 5, we present a very simple proof, due to G. Zapata (see [68]), of Mergelyan's theorem characterizing fundamental weights on the real line. This result was then used by Zapata to show that Hadamard's problem on the characterization of quasi-analytic classes of functions is equivalent to Bernstein's problem on the characterization of fundamental weights.

The result of Chapter 5 are applied in Chapter 6 to $C_o(X;E)$, the space of all continuous functions that are E-valued and vanish at infinity on a locally compact space X, equipped with the uniform convergence topology. We also present here Brosowski, Deutsch and Morris theorem (see [10]) on extreme functionals of the unit ball of the dual of $C_o(X;E)$, generalizing it to vector fibrations.

Analogously, in Chapter 7 we apply the results of Chapter 5 to the space $C_b(X;E)$ of all bounded continuous functions, equipped with the strict topology of Buck. We get both Stone-Weierstrass and Bishop's theorem for this topology. We also characterize extreme functionals of polar set of neighborhoods of the origin of $C_b(X;E)$.

The eighth Chapter deals with the ε-product of L. Schwartz and the approximation property for certain spaces of functions, e.g. Aron and Schottenloher [3] result on the equivalence between the approximation property for a complex Banach space E and the same property for the space of holomorphic mappings on E with the compact-open topology. Also, the proof due to K.-D. Bierstedt [5] of the vector-valued version of Mergelyan's theorem on approximation in the complex plane is to be found in this Chapter. It ends with some results of Bierstedt [6] on the "localization" of the approximation property via maximal anti-symmetric sets.

Chapter 9 deals with nonarchimedean approximation Theory. The first results in this area were proved by J. Dieudonné. He proved in [70], for functions with values in the field of p-adic numbers, the analogues of Weierstrass polynomial approximation theorem, and of Stone-Weierstrass Theorem on density of separating subalgebras. To prove these Theorems he first established the analogues of Tietze's Extension Theorem and his own Theorem on appoximation of functions on cartesian products. In 1949, Kaplansky generalized Dieudonné's Stone-Weierstrass Theorem to the case of functions with values in any field with a (rank one) valuation. (See Kaplansky [72]). The case of arbitrary Krull valuations (or of archimedean valuations other than the usual absolute value of $\mathbb{C}$) was established by Chernoff, Rasala and Waterhouse in [69].

We here treat only the case of rank one, i.e. real valued nonarchimedean valuations. We extend the Dieudonné-Kaplansky Theorem to vector valued functions, more precisely to functions with values in a nonarchimedean normed space over some valued field $(F, |\cdot|)$. Our treatment cover the case of A-modules, where A is an algebra of F-valued functions, and in the case $E = F$ extends Kaplansky's result in the sense that we compute the distance of a function from a module. As a corollary one gets the description of the closure of a module and the density result. We also present Murphy's treatment of vector fibrations in a slightly modified version (see [74]). Results on ideals are also given, extending a result of I. Kaplansky on ideals of function algebras (see I. Kaplansky, *Topological Algebra*, Notas de Matemática Nº 16 (Ed. L. Nachbin), Rio de Janeiro.)

CONTENTS

CHAPTER 1

THE COMPACT-OPEN TOPOLOGY

§ 1 BASIC DEFINITIONS

Throughout this monograph X denotes a non-void Hausdorff space, and E denotes a non-zero locally convex space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). The topological dual of E is denoted by E', and the set of all continuous seminorms on E is denoted by cs(E).

The vector space over $\mathbb{K}$ of all continuous functions taking X into E is denoted by C(X;E). For every non-void compact subset $K \subset X$ and every continuous seminorm $p \in cs(E)$,

$$f \to ||f||_{K,p} = \sup \{p(f(x));\ x \in K\}$$

defines a seminorm on C(X;E). The topology defined by all such seminorms is called the *compact-open topology*.

When E is a normed space, and $t \to ||t||$ is its norm, we write

$$||f||_K = \sup \{||f(x)||;\ x \in K\}$$

for the corresponding seminorm on C(X;E). In particular, when $E = \mathbb{K}$, we write

$$||f||_K = \sup \{|f(x)|;\ x \in K\}$$

and, if no confusion may arise, $C(X) = C(X;\mathbb{K})$.

The vector subspace of all $f \in C(X;E)$ such that f(X) is a *bounded* subset of E, is denoted by $C_b(X;E)$ and topologized by considering the family of all seminorms

$$f \to ||f||_p = \sup \{p(f(x));\ x \in X\},$$

where $p \in cs(E)$. This topology is referred to as the *topology* *of uniform convergence* on X, or as the *uniform topology*.

When X is *compact*, the two spaces $C(X;E)$ and $C_b(X;E)$ coincide, and the compact-open and the uniform topology are the same.

When E is a normed space, and $t \to ||t||$ is its norm, we write

$$||f|| = \sup \{||f(x)||;\ x \in X\}$$

for the corresponding norm on $C_b(X;E)$. If $E = \mathbb{K}$, and no confusion may arise, we write $C_b(X) = C_b(X;\mathbb{K})$.

Given a non-empty subset $S \subset C(X;E)$, we define an equivalence relation on X, by setting, for all $x, y \in X$, $x \equiv y$ (mod. S) if, and only if, $f(x) = f(y)$ for all $f \in S$. Since the elements of S are continuous functions, the equivalence classes (mod. S) of X are closed subsets. The set $S \subset C(X;E)$ is said to be *separating on* X if the equivalence classes (mod. S) of X are sets reduced to points. This is equivalent to say that, for any pair $x, y \in X$ of distinct points, there is $f \in S$ such that $f(x) \neq f(y)$. If S is separating on X, we also say that S *separates the points of* X.

If $K \subset X$ is a *closed* non-empty subset, and $S \subset C(X;E)$, then $S|K$ denotes the subset of $C(K;E)$ consisting of all $g \in C(K;E)$ such that there exists $f \in S$ with the property that $g(x) = f(x)$, for all $x \in K$. In particular, if $K \subset X$ is *compact* and $E = \mathbb{K}$, then $C(K) = C_b(X)|K$, by the Tietze Extension Theorem, when X is completely regular.

It follows easily from the above definitions that for any closed subset $K \subset X$, if $x,y \in K$ then $x \equiv y$ (mod. S) if and only if $x \equiv y$ (mod. $S|K$). Moreover, given any equivalence class $Y \subset K$ (mod. $S|K$) there is a *unique* equivalence class $Z \subset X$ (mod. S) such that $Y = Z \cap K$.

Suppose that E is a *Hausdorff space*, and $S \subset C(X;E)$. Let $A = \{\phi \circ f;\ \phi \in E',\ f \in S\}$. Then for every $x,y \in X$, $x \equiv y$ (mod. S) if, and only if, $x \equiv y$ (mod. A). In fact, the "only if" part is true even when E is not Hausdorff.

§ 2 LOCALIZABILITY

Let A be a subalgebra of $C(X;\mathbb{K})$. A vector subspace $W \subset C(X;E)$ will be called a *module over* A, or an A-*module*, if the function $x \to a(x)f(x)$ belongs to W, for every $a \in A$ and $f \in W$.

Notice that, if B denotes the subalgebra of $C(X;\mathbb{K})$ generated by A and the constant functions, then W is an A-module if, and only if, W is a B-module. Moreover, the equivalence relation $x \equiv y$ (mod. A) is the same as $x \equiv y$ (mod. B).

DEFINITION 1.1 *Let* W $C(X;E)$ *be an* A-*module. We say that* W *is localizable under* A *in* $C(X;E)$ *if the compact-open closure of* W *in* $C(X;E)$ *is the set of all* $f \in C(X;E)$ *such that* $f|Y$ *belongs to the compact-open closure of* $W|Y$ *in* $C(Y;E)$ *for each equivalence class* $Y \subset X$ *(mod. A).*

This is equivalent to say that the compact-open closure of W in $C(X;E)$ is the set of all $f \in C(X;E)$ such that, given $Y \subset X$ an equivalence class (mod. A), $K \subset Y$ a compact subset, $\varepsilon > 0$; and $p \in cs(E)$, there is $g \in W$ such that $p(f(x)-g(x)) < \varepsilon$, for all $x \in K$. We let $L_A(W)$ be the set of all such functions. Notice that $L_A(W)$ is always a closed subset of $C(X;E)$, containing W. It follows that W is localizable under A in $C(X;E)$ if, and only if, $L_A(W)$ is contained in the compact-open closure $\overline{W}$ of W in $C(X;E)$.

Notice too that $L_A(W) = L_B(W)$, if B denotes the subalgebra of $C(X;\mathbb{K})$ generated by A and the constant functions. Thus W is localizable under A in $C(X;E)$ if, and only if, W is localizable under B in $C(X;E)$.

When $E = \mathbb{K}$, every subalgebra $A \subset C(X;\mathbb{K})$ is a module over itself. The definition of localizability is motivated by the classical Stone-Weierstrass Theorem. Indeed, we have the following result which connects the notion of localizability with the usual statement of the Stone-Weierstrass Theorem. (See Theorem 1, § 17, Nachbin [43]).

PROPOSITION 1.2 *Let* $A \subset C(X;\mathbb{K})$ *be a* $\mathbb{K}$*-subalgebra, and let* $f \in C(X;\mathbb{K})$. *Then* $f \in L_A(A)$ *if, and only if, the following two* *conditions are satisfied:*

(1) *for every* $x \in X$ *such that* $f(x) \neq 0$, *there exists* $g \in A$ *such that* $g(x) \neq 0$;

(2) *for every* $x, y \in X$ *such that* $f(x) \neq f(y)$, *there exists* $g \in A$ *such that* $g(x) \neq g(y)$.

PROOF (a) Suppose $f \in L_A(A)$. Let $x \in X$ be such that $f(x) \neq 0$. Assume that $g(x) = 0$ for all $g \in A$. Let $Y \subset X$ be the equivalence class (mod. A) that contains x, and let $K = \{x\}$. By hypothesis, there is $g \in A$ such that $|f(x) - g(x)| < \varepsilon = |f(x)|$. Since $g(x) = 0$, this is a contradiction. Therefore (1) is satisfied. The proof that (2) is satisfied is analogous, so we omit the details.

Suppose now conditions (1) and (2) are satisfied. Let $Y \subset X$ be an equivalence class (mod. A). By (2), f is constant on Y. Let $u \in \mathbb{K}$ be its constant value. If $u = 0$, then $g = 0 \in A$ coincides with f on Y. Assume now that $u \neq 0$. By (1), there is $g \in A$ such that $g(x) \neq 0$, where $x \in Y$ is an arbitrary point fixed in Y. Then $g(y) = g(x)$ for all $y \in Y$. Therefore $h = (u/g(x))g$ belongs to A and $h(y) = u = f(y)$ for all $y \in Y$. Hence $f \in L_A(A)$.

§ 3 PRELIMINARY LEMMAS

In this section we shall obtain two lemmas that will be useful in the "approximate partition of unity" needed in the proof of the main theorem of this chapter. The second of those lemmas is due to Jewett [32], who employed it in his proof of a variation of the Stone-Weierstrass theorem. It is a corollary of the classical Weierstrass polynomial approximation theorem, but we prefer to present Jewett's direct proof, to make our version of the Stone-Weierstrass theorem independent of Weierstrass theorem.

LEMMA 1.3 *Let* $A \subset C_b(X;\mathbb{R})$ *be a subalgebra containing the constants, and let* $Y \subset X$ *be an equivalence class (mod. A). For every* $\varepsilon > 0$, *and every open subset* $U \subset X$, *containing* Y, *such that the complement of* U *is compact, we can find* $g \in A$ *such that* $0 \leq g \leq 1$, $g(y) = 1$ *for all* $y \in Y$, *and* $g(t) < \varepsilon$ *for* $t \notin U$.