Mathematics Research Developments

Fractional Calculus

Applications

Roy Abi Zeid Daou
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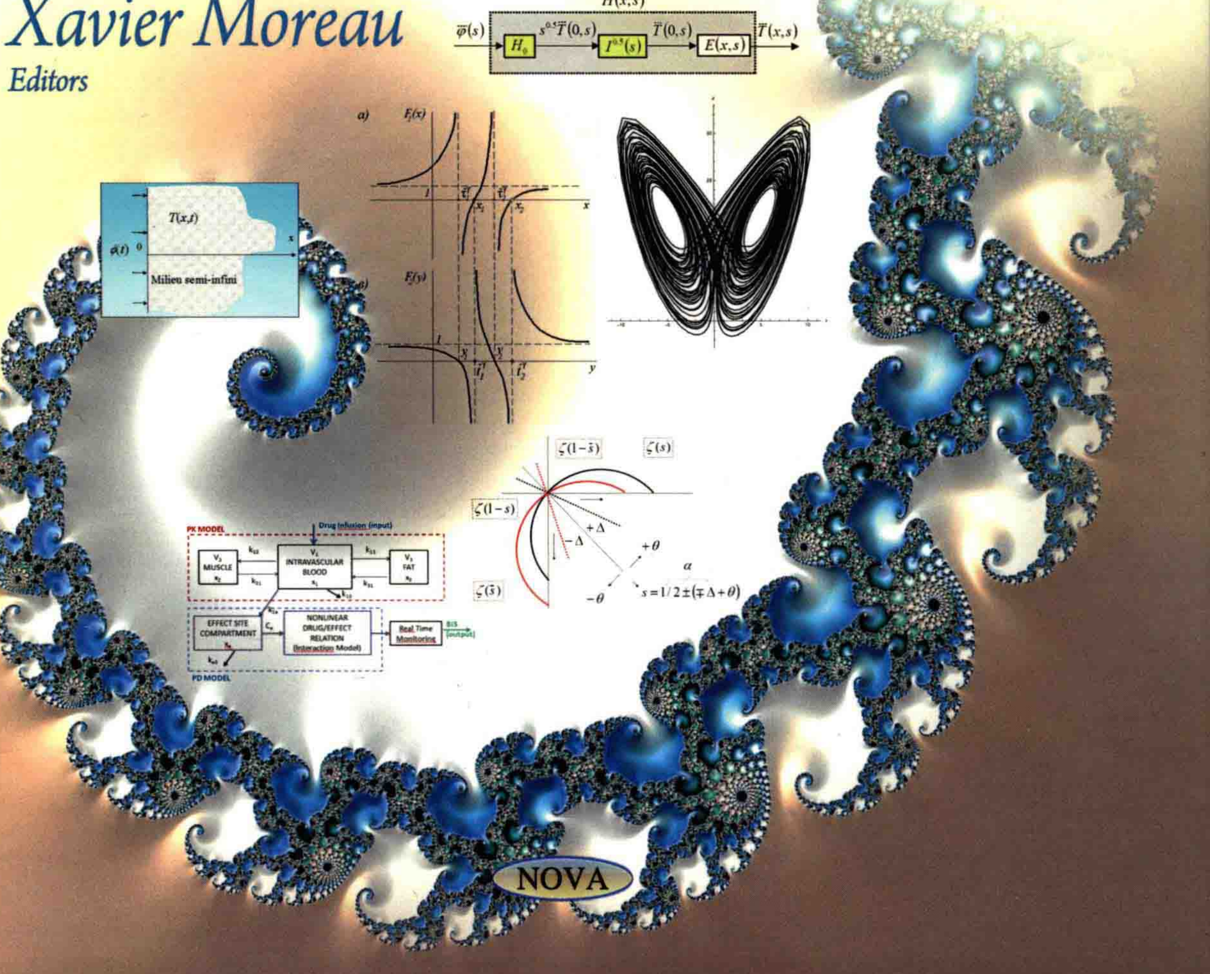
Editors



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ROY ABI ZEID DAOU
AND
XAVIER MOREAU
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PREFACE

After presenting the first volume of this two-volume book, presenting a lot of mathematical and theoretical studies and research related to non-integer calculus, the second volume illustrates lots of applications related to this domain. In the following, we will present the applications discussed in this book.

This volume is made up of 11 chapters. The first chapter presents the heuristic power of the non-integer differential operators in physics starting from the chaos to the emergence, the auto-organizations and the holistic rules. The second chapter shows the dynamics of the fractional order chaotic systems along with some applications. The third chapter represents the pressure control of gas engines by non-integer order controllers by showing a novel trend in the application of the fractional calculus to automotive systems.

Chapter 4 shows the way to model fractional order equations using state space modeling along with some applications. Another application related to this domain is the thermal diffusive interface. Chapter five shows the analysis of a semi-infinite diffuse plane medium along with the equations that model this medium, and some frequency and time domain responses. However, chapter six treats this problem by controlling this plant using the well-known CRONE controller.

Chapter eight presents the adaptive second-order fractional sliding mode control with an application to a water tanks level system. Chapter nine treats the mechanical aspect by showing the features of the fractional operators applied to this domain. Also, chapter nine presents the theory of diffusive stresses based on the fractional advection-diffusion equation.

The modeling of drug diffusion during general anesthesia using Fractional Calculus is shown in chapter ten and is considered as another application related to the biomedical field. At the end, chapter eleven represents an overview of the fractional fuzzy controllers by showing the analysis, the synthesis and the implementation of this module.

To sum up, this second volume presents lot of applications of fractional calculus in several engineering domains as the thermal, the automotive, the mechanical, the biomedical and much more. Note that this volume was preceded by a first volume that focuses on the mathematical and theoretical aspects of fractional calculus.

FOREWORD

Non-integer differentiation does not escape to the slogan "different operator, different properties and performances". This is indeed the concise formula that is likely to explain the "why" of this operator, especially as most of its properties and performances favorably distinguish, not only the operator itself, but also the models that use it.

It is true that we have established non-integer models that overcome the mass-damping dilemma in mechanics and the stability-precision dilemma in automatic control, the technological achievements associated with these models have been made possible thanks to an adequate synthesis of the non-integer differentiation operator.

It is indeed the idea to synthesize non-integer differentiation (in a medium frequency range) through a recursive distribution of passive components, of transitional frequencies or of zeros and poles, which is at the origin of the non-integer differentiation operator real-time use and, therefore, of both analogical and numerical applications that arise from it. As for the corresponding dates, the synthesis as we have led it has been developed by stages and thus proposed and experimented in the 70s for half-integer orders, in the 80s for real non-integer orders and in the 90s for complex non-integer orders.

The first technological applications of this operator (henceforth usable in real-time) and notably the first application in 1975 of a "half-integer order controller" to the frequency control of a continuous dye laser, have widely contributed to take the non-integer differentiation out of the mathematician drawers and to arouse new developments likely to enrich the theoretical corpus of circuits and systems.

In this way France has been the first country to know a renewed interest on non-integer differentiation, this renewal having been well relayed thanks to the dynamism of the foreign scientific communities, at the European level as well as at the international level.

In this context, the French institutions have encouraged research in this field through the acknowledgement of major scientific advances and the support of initiatives or actions aiming to favor the synergies between the different themes and between the academic and industrial components, the University-Industry partnership having indeed been nationally rewarded by the AFCET'95 Trophy distinguishing the CRONE suspension as best technological innovation. Concerning the acknowledgements, let us cite the selection of the CRONE control as a "striking fact" of the Centre National de la Recherche Scientifique (CNRS) in 1997 and as "Flagship Innovation" of Alstom in 2000 (Hanover and Baden Baden International Fairs, 2000), a Silver Medal of the CNRS in 1997 and the Grand Prix Lazare Carnot 2011 of the Science Academy (founded by the Ministry of Defense). Concerning the supports, let us cite the actions financially supported by the CNRS and the Ministry of Research: the edition of "La commande CRONE" (Hermès, 1991) with an exceptional help of the ministry; the International Summer School "Fractal and hyperbolic geometries, fractional and fractal derivatives in engineering, applied physics and economics" (Bordeaux, 1994); the national project of the CNRS, "Non-integer differentiation in vibratory insulation" (1997-1999); the colloquium "Fractional differential systems" (Paris, 1998); the launching in 1999 of the thematic action of the Ministry of Research "Systems with non-integer derivatives"; the launching in 2004 of the IFAC Workshop "Fractional Differentiation and its Applications" through the first Workshop FDA'04 (Bordeaux, 2004) with S. Samko as chairman of the International Program Committee ; the magisterial lecture "From diversity to unexpected dynamic performances" initiated by the French Science Academy (Bordeaux, 5 January 2012).

But this academic support also found a guarantee in the industrial support brought by a strong partnership with major companies as PSA Peugeot-Citroën, Bosch (Stuttgart) and Alstom, such a partnership having indeed led to a high number of patents and technological transfers that have widely proved the industrial interest of non-integer approaches.

Alongside the shared efforts to inscribe these approaches in the realist frame of the University-Industry relations, our efforts have never stopped being shared with those of the international scientific community to develop the best relations and collaborations within this community. Without aiming for exhaustiveness, let us cite the involvement of European countries in the diffusion, promotion and animation within the community: the research group "Fracalmo", which originates from **Fra**ctional **cal**culus **mo**delling, started with a round table discussion in 1996 during the 2nd International Conference "Transform methods and special functions" held in Bulgaria; the journal "Fractional Calculus and Applied Analysis" (FCAA) started in 1998 with V. Kiryakova as managing editor; the survey on the "Recent history of fractional calculus" (*Communications in Nonlinear Science and Numerical Simulation*, 2010) at the initiative of J.T. Machado who desired to make an inventory of the major documents and events in the area of fractional calculus that had been produced or organized since 1974; the symposium "Fractional Signals and Systems" (FSS) launched by M. Ortigueira in 2009 at Lisbon, then held in 2011 at Coimbra (Portugal) and in 2013 at Ghent (Belgium). Both in and out of Europe, let us also recall the various events of the Workshop FDA after its launching at Bordeaux in 2004 (under the aegis of IFAC): Porto (Portugal) in 2006; Ankara (Turkey) in 2008; Badajoz (Spain) in 2010; Nanjing (China) in 2012; Grenoble (France) in 2013; Catania (Italy) in 2014 (under the aegis of IEEE).

Founder of the CRONE team that counts today about ten permanent researchers, I recognize this team to have always escorted me in the federative actions, launched within the national or international scientific community, to energize and harmonize the researches on both theoretical and applicative aspects of non-integer differentiation.

Xavier Moreau, member of the CRONE team, and Roy Abi Zeid Daou, associated member, who are implied in this book as coordinators well attest this will to be at the initiative of collective actions. It is then a pleasure for me, through this foreword, to warmly congratulate them for their efforts of scientific coordination at the international level, especially as the number of proposed contributions in this book well proves the dynamics of the fractional (or non-integer) community.

Alain Oustaloup

CONTENTS

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Chapter 1

THE HEURISTIC POWER OF THE NON INTEGER DIFFERENTIAL OPERATOR IN PHYSICS: FROM CHAOS TO EMERGENCE, AUTO-ORGANISATIONS AND HOLISTIC RULES

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Abstract

The use of fractional differential equations raises a paradox due to the non-respect of the space time noetherian axioms. In environments characterized by scaling laws (hyperbolic geometry associated with fractional diff-integral) energy is no more the invariant of the dynamics. Nevertheless the experimental action requiring the use of energy, the relevant representation of the fractional-process, must be extended. The extension is carried out using the canonical transfer functions in Fourier space and explained by their links with the Riemann zeta function. Category theory informs the extension problem.

Ultimately the extension can be expressed by a simple change of referential. It leads to embed the time in the complex space. This change unveils the presence of a time singularity at infinity.

The paradox of the energy in the fractality illuminates the heuristic power of the fractional differential equations. In this mathematical frame, it is shown that the dual requirement of the frequency response to differential equations of non-integer order and of the notherian constraints make gushing out a source of negentropique likely to formalize the emergence of macroscopic correlations into self-organized structures as well as holistic rules of behaviour.

1. Introduction

The heuristic power of the fractional derivation operator is based at least upon three theoretical foundations

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- the canonical link between the fractional operators and fractal geometries. This type of operator expresses the non-differentiability of the geometry (Le Méhauté, 1982a),
- the generalization of exponentiation operation in the frequency space. This generalization is related to partial hyperbolic geodesics (Nivanen, 2005),
- the relation between this generalization and Riemann's (and Goldbach's) hypothesis that gives meaning to the universal approximations of all analytic functions (Voronin, 1975) (Bagshi, 1987).

The assertion of heuristic power of fractional diff-integral operators finds its sources, within the first generalization of Zipf laws in the form of so-called Zipf-Mandelbrot law (Le Méhauté, 1974, 1977). These publications link the recursive structures of the statistical supports (trees and hyperbolic groups) and the thermodynamic laws by merging geometries and singularity distributions. By extending the concept of recursivity into geometry, Mandelbrot published in 1975 his first works on 'fractal geometries' (Mandelbrot 1975, 1982), a word forged for the purposes of the case. To describe the exchange of energy and matter dynamics in heterogeneous environments (ie fractal structures) (here considered as a mathematical distribution ground (Schwartz, 1950)) the author of this paper has developed at the end of the seventies a model of transfer of energy on fractal structure embedded in a physical environment (electrolyte, liquid, air, etc.) named TEISI model (Le Méhauté, 1980, 1982a, 1984). This model merges a class of convolution between fractal structures (geometric factor) and traditional differential operators (dynamics factor). It gives a theoretical base to a new type of irreversible dynamics. After the Oldham and Spanier (Oldham, 1974) and Nigmatullin's senior formal works (Nigmatullin, 1975), the TEISI model and its fractional operators feature the way for a theoretical analysis of processes characterized by an irreversibility based upon geometry. Jointly, the relevance of these operators for the optimization of electrochemical energy storage devices (Le Méhauté, 1982b, 1986) was performed. There followed a 'declension' of many issues about irreversibility that leads to the establishment of a link between fractals and hyperbolic geometries (Nivanen, 2005), number theory (Le Méhauté, 2010) and theory of categories (Riot, 2013).

Despite an undeniable prospective effectiveness and numerous coherent experimental data (Le Méhauté, 1983, 1986, 1992), opinion still mainly given to this work, is the one expressed by Pierre Gilles de Gennes referee to the Academy of Sciences of Paris, in the defense of the first publications of the author (Le Méhauté 1982a). He wrote in footnote page "*L'introduction d'une hypothèse de self similarité sur la structure de certaines électrodes parait extrêmement féconde. Par contre l'équation reliant directement la dépendance en fréquence de l'impédance à un paramètre purement géométrique (...) semble très conjecturale. Dans les quelques exemples connus de transport impliquant un fractal (amas de percolation, réseaux de Sierpinsky,...) l'exposant de conductivité n'est pas lié directement à la dimensionnalité*". Let us see why the assumption of the *transport of species* evoked by P. G. de Gennes, twists the empirical point of view with respect to the hypothesis based on the *transfer of energy*. Due to the range of frequency analysis (Hertz plus microwaves range) the analysis of the efficiency of (i) the batteries and (ii) of the cable impedance served as an excellent compass.

Let us see why despite the approval of the scientific community to de Gennes' reserves, and after the epistemological choice of this community for considering the fractal interface only as an external boundary conditions with regard to the transportation of physical species (matter, ions, electrons, entropy, energy), the author continued his work in the original direction by addressing the issue of *transport in fractal* media through the theory of distributions as a simple limit case of *transfer process*.

2. A Nice Equation for an Heuristic Power

"Sole a beautiful equation may have a deep meaning!" This is the case of the equation below. It establishes a merging between fractal geometry and the operator of the fractional differentiation in the Fourier space, via the set of the integer numbers:

$$[\eta(\omega)]^d (i\omega\tau) = const \ , \tag{1}$$

As shown in (Le Méhauté, 1982a), this equation does not *a priori* assume, (as one may understand *a posteriori*) the presence of a link between space and time. In the frame of distribution theory it just assumes the empirical constraint for a measurement of a recursive ordered structure (interface), which canonical expression is given via a set of integers. The emergence, a posteriori, of temporal variables is a simple mathematical consequence of the use of this one dimensional set when it is plunged in the set of pure complex numbers. This mathematical embedding comes from the interpretation of the Hausdorff-Mandelbrot equation

$$v\eta^d = const \ , \tag{2}$$

(Mandelbrot, 1975, 1982) by using the number expressed by. In equation (1) the complex number 'i' expresses the independence of the local measure of the space with regards to the computation required by a global normalization of the fractal interface. The reference 'τ' expresses the requirement for time normalisation. The normalization keeps the scaling invariance to the fractal geometry, when it is used as support of the exchange for the transfer of energy.

Beyond all the outstanding issues about the definition of the operator in time (the whole of this book), this equation expresses the issue open by the non-integer derivation, in its recursive simplicity. For the reasons given above and developed below, we assert that the Fourier space is the natural space of description for all recursive geometries:

$$\eta(\omega) \approx \frac{1}{(i\omega\tau)^{1/d}} \ . \tag{3}$$

We maintain that, including in a majority of experimental cases, it exists a direct link between the expression of the derivation of non-integer order in this space and the fractal dimension of a substructure that is the support of the transfer of energy. It is expressed physically through a generalization of the concept of capacity that we called 'Fractance' (i.e., capacity on fractal support).

This concept founds the use of non-integer derivative in physics. Moreover the use of this formulation in the frequency domain (Fourier or Laplace) postulates a 'computational foundation' for the emergence of well-ordered variable that we name the 'irreversible time' which is practically useless in mechanics (Rovelli, 2004, 2006) (Connes 1994)

Such a variable must be considered like simple mathematical object in accordance with Hausdorff-Mandelbrot rule. In practice, equation (1) interpolates an invariant factor of the fractal dynamics between the concept of velocity (, $d = 1$) and concept the diffusion coefficient (, $d = 2$). Probably intuitively perceived by de Gennes when assuming exclusively *transport* processes, this interpolation points out an apparent physical 'weakness'. Indeed, due to its dimensions, the new invariant cannot possess a simple physical meaning. Meanwhile at the beginning of the eighties, the heuristic hypotheses of dynamic recursivity described via diff-integral, was already well assured by the results obtained from the efficiency of the batteries, and also supports preliminary development of the CRONE control (Oustaloup, 1983).

There was something of a contradiction. Including with the Mandelbrot's works the engineering was at that time the spearhead of dynamics in fractality and advanced open interrogations. The paradox raised by the contradiction between theory and experiment, (no solved by the use of the concept of transport) unveiled an unquestionable opportunity for making a conceptual breakthrough.

3. SWOT Method, Non Integer Diff-Integral and Co-Dimension

We recall that the TEISI model assumes a 2D symmetry of the macroscopic experimental complex structure which geometry looks like a capacitor. In addition the symmetry of the force-field locates the fractal dimension d in the range given by.

The relevant remark of P. G. de Gennes concerning the weakness of the TEISI model was finally upset as follows: as a matter of fact, the relevant dimensional invariants required in expression (1) are not part of the usual invariants used when energy is exchanged upon integer dimensional interface. In that standard case, the dimensional equation for the energy is:

$$E = ML^2t^{-2} \ , \tag{4}$$

(in Fourier space) and the dimension of the physical action is given by

$$A = ML^2t^{-1} , \tag{5}$$

(in Fourier space). These dimensions cannot be simply applied when environments is characterized by scaling properties. It has to be really 'adapted' to the problem. The standard energy is expressed in space-time by the square of a velocity. This square factor cannot appear naturally when dynamics is deployed in fractal geometry: indeed, the support is not differentiable and therefore, cannot easily have any gradient operator at its disposal.

The second question addressed by the problematic concerns the global point of view upon the interface: the space-time integral, expressed by the macroscopic law

$$x^d \approx t \ , \tag{6}$$

cannot be reduced to the diffusive formulation

$$x^2 \approx t \ , \tag{7}$$

or even to

$$x^{d/2} \approx t \ , \tag{8}$$

initial de Gennes proposal. Because, any experimental process is dissipative it might appear simple to express the problem of fractality as an anomalous diffusive process

$$[\eta(\omega)]^2 (i\omega\,\tau)^{1/d} = const\,. \tag{9}$$

Alas! There was worse to come: these formulations didn't solve, in any manner, the controversial issue *transport* against *transfer* pointed out above. Are these threats, a condemnation of the use of distribution theory and the non-integer order operators? The answer is clearly negative. All attempts to simply mask the defect of fitting between the concept of energy and the concept of fractal, is just a dead end. This question requires, as we have shown very early (1978), a reference to the concept of fractal co-dimension (Tricot, 1999). If the formulation (1) infringes the principles of Emmy Noether (energy cannot more be the expression of the homogeneity of fractal space-time), the difficulties are not mathematically intrinsic but extrinsic. They are strongly related to the contradiction between the Noether's axioms and the scaling properties of the fractal interface measurements. The contradiction between standard theories and experiments, when fractal media is involved, is due to the restrictive requirement imposed by the use of energy for checking the physical properties. This difficulty must be addressed to an epistemological issue that not only assumes an extrinsic meaning of energy (homogeneity of space-time) but uses of balls (in maths) or/and energy (in physics) to perform all practical measurement even if far from homogeneity (complex media). The non-differentiability and the related fractional statistics, hidden behind the transfer TEISI model, unveil precisely the conceptual limitations that must be addressed head on.

If the fractal dimension is *d*, let us observe that the Noetherian homogeneity can be conceptually 'restored' using the concept co-dimensional space by writing

$$E_{\omega} = \left(ML^d t^{-1}\right)\left(L^{2-d}\right)\,, \tag{10}$$

or, in Fourier space, the geometric co-density of energy

$$E_{\omega} \,/\, L^{2-d} = \left(ML^d t^{-1}\right)\,, \tag{11}$$

is obviously the Hausdorff-Mandelbrot dimensional content of the co-fractal space. Let us also observe that the formulations can be reversed by writing

$$E_{\omega} = \left(ML^{2-d}t^{-1}\right)\left(L^{d}\right) \qquad (12)$$

and

$$E_{\omega} / L^{d} = \left(ML^{2-d}t^{-1}\right). \qquad (13)$$

This inversion points out one of the major symmetry for the transfer of energy in fractal media. The question raised by these formulas is the meaning of the co-fractality (). It is obviously related to the embedding of the fractal interface in the two dimensional space. This embedding is required to perform measurement and experiences as well. As exemplified for a long time in energy storage industry (battery energy/power balance optimization), we can confirm that the fractality, which raises the paradox Energy vs Fractal, also opens up new opportunities for technical breakthrough: the design of the geometry is an extrinsic optimization factor of the irreversibility of the process deployed in fractal media. Energy storage requires 3D structures whereas Power requires 2D interface. Depending of the need, the fractal dimension balance is obviously in between. Conversely, the entropy/negentropy ratio must control the metric and therefore 'curvature' of a fractal geometry.

We will see that the constraint of continuity along the fractal object is a central factor for optimization principle. It is expressed by a geometrical phase. Then the metric of the geometry determines the ratio of the entropy/negentropy production. To easily understand the contents of the role of this ratio we need to consider briefly the fractional generalization of exponentiation.

4. The Generalization of the Exponential Concept

The 'exponential' and 'logarithm' functions play a central role in physics. This role has its origin in the morphism between addition and multiplication. As shown in category theory, this morphism is essential in mathematics and physics because it establishes a strict relationship between the constructivism performance, that is the addition of ordered structures (adding is considered then as a co-product), and the partition or deconstruction of the same structures into substructures (represented by the product). The question addressed by this morphism concerns the content of the 'differance' with 'a' and not 'e' (Derrida, 1973) which controls the irreversibility. We shall find in non-extensive thermodynamic developments, a useful analytic extension of exponential functions and logarithms within the time space (Nivanen, 2005). However, these extensions can only reach a deep geometrical meaning in Fourier space.

The exponential term in the Fourier space is always given through a semicircle

$$Z(\omega\tau) \approx \frac{R}{1 + i\omega\tau}\ . \qquad (14)$$

This is a transformation obtained by a geometrical reversing of a straight line with respect to a point located outside of the line. This modular formulation (Ghys, 1990) is canonical and it is rightly adapted to our purposes because it opens the matter of geodesics in hyperbolic space. The impedance takes the dimension of a resistor:. For longtime and *R* defines a state in the thermodynamic sense. The semicircle, inverse of a straight line, is also a geodesic because its inverse Fourier transform is (i) the classical first order relaxation dynamics solution and (ii) the merging of additive and multiplicative properties. Both properties give to the exponential function a canonical characteristic of a closure (construction, ω-algebra/τ-algebra, deconstruction).

Starting from that observation an heuristic approach built from this canonical form leads to consider the TEISI model as a rather natural ‘declension’ of exponential function because the introduction of the fractal geometry as an interface for transfer of energy involves geodesics as arcs of circle. We recall that TEISI model leads

$$Z_{1/d}(\omega\tau) \approx \frac{R\cos\varphi}{1+(i\omega\tau)^{1/d}} \quad . \tag{15}$$

Then, we can confirm that

$$\eta(\omega\tau) = \frac{u(\omega\tau)}{v(\omega\tau)} \quad , \tag{16}$$

expresses the hyperbolic distance on . The scaling correlations conditions are deliver for (respectively) where is the phase angle associated with the rotation given by in the complex plan. This phase angle which requires information coming from the environment renders the above Cole and Cole representation incomplete. The paradox is not only the absence of standard thermodynamic equilibrium state function at infinity, (when this infinity is defined by), but also the requirement for a couple of referentials to define the whole representation. Even if the origin of the phase at infinity comes from intrinsic recursive correlations characterizing the fractal support of the process, it requires the embedding within a larger environment. This requirement looks like Kan extension and virtualisation of sets in the frame of the theory of categories (Riot, 2013). In spite of these difficulties Gemant analysis (Gemant, 1935) and Cole-Cole synthetic formulation (Cole, 1942) has a real empirical effectiveness. Let us observe that the fractional process is characterized by. Then the points at infinity contain phase information which reduces relatively to *R* the entropy. Therefore, beyond the questions raised by the temporal expression, the Fourier space representation reveals a new core of problems concerning the balance between entropy and negentropy production in complex media: the role of the boundary (environment). Then the question of local causality in the traditional sense can be clearly addressed precisely from the incomplete TEISI transfer formulation.