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Comprehensive Studies
in Mathematics

Constructive Approximation

Advanced Problems

构造逼近

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A Series of Comprehensive Studies in Mathematics

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With 10 Figures

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Preface

In the last 30 years, Approximation Theory has undergone wonderful development, with many new theories appearing in this short interval. This book has its origin in the wish to adequately describe this development, in particular, to rewrite the short 1966 book of G. G. Lorentz, "Approximation of Functions." Soon after 1980, R. A. DeVore and Lorentz joined forces for this purpose. The outcome has been their "Constructive Approximation" (1993), volume 303 of this series. References to this book are given as, for example [**CA**, p. 201].

Later, M. v. Golitschek and Y. Makovoz joined Lorentz to produce the present book, as a continuation of the first.

Completeness has not been our goal. In some of the theories, our exposition offers a selection of important, representative theorems, some other cases are treated more systematically. As in the first book, we treat only approximation of functions of one real variable. Thus, functions of several variables, complex approximation or interpolation are not treated, although complex variable methods appear often.

Most of the chapters of the present book can be read independently of each other. They fall into groups: Chapters 1–6 deal with polynomial and spline approximation – in some sense they continue the themes of [**CA**]. Chapters 7–10 contain a fairly complete theory of rational approximation. Chapters 12–14 treat widths and entropy of classes of functions. But even within the groups, chapters are more or less independent, except that it is advisable to read Chapter 3 before Chapter 4, while Chapter 7 is indispensable for Chapters 8 and 10, and Chapter 13 for Chapter 14. Most of the information about Banach function spaces needed in the two volumes of **CA** can be found in [**CA**, Chapter 2], in §7 of Chapter 1 of the present volume, and in the book of Bennett and Sharpley [B-1988]. We also provide a quick new look at some of the important approximation theorems: for polynomials in §7 of Chapter 1, for splines in §1 of Chapter 6.

Related branches of Analysis: Fourier Series, Orthogonal Polynomials, Potential Theory, Functional Analysis, even Number Theory are our allies. We use their methods; some of the needed results are collected for the reader in the four Appendices.

For the development of the Approximation Theory, one cannot be sufficiently thankful to the Russian (Soviet) mathematicians: to Chebyshev, A. A. Markov, Bernstein, Kolmogorov and others, who built its foundations.

At present Approximation Theory is popular worldwide, with the new theories of splines, of rational approximation, of wavelets.

We are very grateful to A. A. Pekarskii (Grodno, Belarus), who has prepared for us Chapter 10, which deals with complex methods in rational approximation. Our colleagues, Berens, R. A. Lorentz, Stahl, Erdélyi, Lubinsky, Totik have helped us with concrete problems. We are also indebted to Blatt, Buslaev, Chui, Jetter, Maiorov, Shechtman, Varga and others for useful advice. Margaret Combs at the Department of Mathematics, The University of Texas, has very ably typed many chapters of the book.

The book has an extensive bibliography, which can also serve as Author's Index. Each quoted journal article is followed by the number of page, where it is referred to in the text. There is also a Subject Index.

The authors would be grateful for any comments or proposals of corrections from the readers.

The Authors

Contents

Chapter 1. Problems of Polynomial Approximation

This chapter contains a discussion of some important problems of approximation, mainly by algebraic polynomials. We begin with properties of polynomials of best approximation: some examples in §1, distribution of their alternation points on the interval in §2, distribution of their zeros in the complex plane in §3. In §4, as an exception, we discuss approximation by entire functions, and the error of approximation in Banach spaces. In §§5–6, we give a solution of a problem of Bernstein, about the weighted polynomial approximation on $(-\infty, \infty)$. Spaces for approximation problems are found in §7.

§ 1. Examples of Polynomials of Best Approximation

Polynomials of best uniform approximation on the circle $\mathbb{T}$ or an interval $[a, b]$ are described by the theorems of Chebyshev (see, for example, [**CA**, Theorem A, p.58, or Theorem 5.1, p.74]). Only in exceptional cases can they be given explicitly. Here are some examples.

Let $n_1, n_2, \ldots$ be odd integers ≥ 3, we write $N_k = n_1 n_2 \cdots n_k$.

Theorem 1.1. *Let f be a continuous function on $\mathbb{T}$, with the Fourier series of one of the forms*

$$a_0 + \sum_{k=1}^{\infty} a_k \cos N_k t \quad , \quad \sum_{k=1}^{\infty} b_k \sin N_k t \ . \tag{1.1}$$

Then the trigonometric polynomials of best approximation to f are precisely the partial sums of (1.1). *(In particular, the series converge uniformly.)*

Proof. Consider for example, the first series (1.1). The statement of the theorem asserts that the partial sum $S_{k-1}(t) = \sum_{j=0}^{k-1} a_j \cos N_j t$ is the best approximation to f, among all polynomials of degree $\leq n$, for each $n = N_{k-1}, \ldots, N_k - 1$. The difference $R_k = f - S_{k-1}$ has the following properties. Its Fourier series is $\sum_{j=k}^{\infty} a_j \cos N_j t$, and by Fejér's theorem, R_k lies in the closed span of these cosines. Hence R_k has period $2\pi/N_k$. In addition, since all N_j are odd, R_k is odd about the center $c := c_\ell$ of each of the intervals

$$I_\ell = \left[\frac{\pi}{2N_k} + \frac{2\pi\ell}{N_k} \quad , \quad \frac{\pi}{2N_k} + \frac{2\pi(\ell+1)}{N_k}\right] \quad , \quad \ell = 0, \ldots, N_k - 1 \ ,$$

that is, it satisfies $R_k(c-t) = -R_k(c+t)$. It follows that the absolute maximum M of R_k on $\mathbb{T}$ and its absolute minimum $-M$ are taken on each I_ℓ at points symmetric about c_ℓ. We get enough alternation points to apply Chebyshev's theorem. Similarly for the second series (1.1). □

If the coefficients a_k in the first series (1.1) are of the same sign, we can obtain an explicit formula for the error of approximation $E_n(f)$. We explain this for the algebraic case. Let $a_k \geq 0$, $\sum a_k < +\infty$ and let f on $[-1,1]$ be given by

$$f = \sum_{j=1}^{\infty} a_j C_{N_j} , \tag{1.2}$$

where C_n are the Chebyshev polynomials [**CA**, §6, Chapter 3]. By the standard substitution $x = \cos t$ and Theorem 1.1, $S_n(x) = \sum_{j=1}^k a_j C_{N_j}(x)$, $N_k \leq n < N_{k+1}$ is the polynomial of best approximation to f from $\mathcal{P}_n$ and

$$E_n(f) = f(1) - S_n(1) = \sum_{j=k+1}^{\infty} a_j , \qquad N_k \leq n < N_{k+1} , \ k = 1, 2, \ldots . \tag{1.3}$$

Here is another concrete example, known already to Chebyshev.

Theorem 1.2. *Let* $f(x) = (x-a)^{-1}$, $x \in [-1,1]$, *where* $a > 1$. *Then for* $c := a - \sqrt{a^2-1} < 1$,

$$E_n\left((x-a)^{-1}\right) = \frac{4c^{n+2}}{(1-c^2)^2} . \tag{1.4}$$

Proof. The formula $x = \frac{1}{2}(w + w^{-1})$ defines a one to one map of the complex x-plane split by $[-1,1]$ onto the disk $|w| < 1$. To each $x \in [-1,1]$ correspond two values of w on $|w| = 1$, related by $w_1 = w_2^{-1}$. (See [**CA**, §2, Chapter 4].) Let $0 < c < 1$ be given by $a = \frac{1}{2}(c + c^{-1})$, that is, by $c = a - \sqrt{a^2-1}$. Then

$$\Phi(x) = \frac{M}{2}\left(w^n \frac{c-w}{1-cw} + w^{-n}\frac{1-cw}{c-w}\right) \tag{1.5}$$

defines a function on $\mathbb{C}$. We note that $w^k + w^{-k}, k = 0, 1, \ldots$ is a polynomial in x of degree k and that

$$(1+c^2)(1-\frac{x}{a}) = (\frac{1}{w} - c)(w - c) .$$

Therefore

(1.6)
$$\begin{aligned}\Phi(x) &= -\frac{M}{2}\left\{w^{n-1}\frac{w-c}{w^{-1}-c} + w^{1-n}\frac{w^{-1}-c}{w-c}\right\} \\ &= -\frac{M}{2}\left\{w^{n-1}(w-c)^2 + w^{1-n}(w^{-1}-c)^2\right\}\left(1-\frac{x}{a}\right)^{-1}(1+c^2)^{-1} .\end{aligned}$$

We see that $\Phi(x)$ has the form

$$\Phi(x) = \frac{A}{x-a} - P_n(x) \ ,$$

where P_n is a polynomial of degree n with real coefficients. Since

$$A = \lim_{x\to a}(x-a)\Phi(x) = \frac{M}{2}\lim_{w\to c}\frac{(w-c)(cw-1)}{2cw}w^{-n}\frac{1-cw}{c-w} = \frac{M(1-c^2)^2}{4c^{n+2}} \ ,$$

we select $M = 4c^{n+2}(1-c^2)^{-2}$, and have then

$$\Phi(x) = \frac{1}{x-a} - P_n(x) \ .$$

As w moves on the upper semicircle $|w| = 1$ counterclockwise, x moves on $[-1,1]$ from 1 to -1. By (1.5), $\Phi(x) = \frac{M}{2}(\Psi(w) + \Psi(w)^{-1})$, where $\Psi(w) = w^n(c-w)(1-cw)^{-1}$. Since $|\Psi(w)| = 1$ on $|w| = 1$, we have $|\Phi(x)| \le M$, $|w| = 1$. The function Ψ has $n+1$ zeros inside $|w| = 1$, and because of symmetry, $\arg\Psi(w)$ changes from π to $(n+2)\pi$ on the upper semi-circle. If $\arg\Psi(w) = k\pi$, $k = 1,\dots,n+2$, then $\Phi(x) = M$ or $\Phi(x) = -M$ for even or for odd k, respectively. By Chebyshev's theorem on $[-1,1]$, P_n is the polynomial of best approximation for $(x-a)^{-1}$, and the error of approximation is M. □

Let P_n, P_{n+1} be two polynomials of best approximation to $f \in C[-1,1]$, and let $P_n \ne P_{n+1}$. Then:

(1.7) $Q := P_n - P_{n+1}$ has $n+1$ distinct zeros in the open interval $(-1,1)$.

Indeed, let $-1 \le x_1 < \cdots < x_{n+2} \le 1$ be $n+2$ alternation points for P_n (from [**CA**, Theorem 5.1, p.74]). If for instance $f(x_j) - P_n(x_j) > 0$, then

$$f(x_j) - P_n(x_j) = \|f - P_n\| > \|f - P_{n+1}\| \ge f(x_j) - P_{n+1}(x_j) \ , \tag{1.8}$$

so that $Q(x_j) < 0$. Similarly $Q(x_{j+1}) > 0$. Thus, Q changes sign on each of the intervals $[x_j, x_{j+1}]$.

Can it happen that all polynomials of best approximation to $f \in C[-1,1]\setminus \mathcal{P}$ have a common zero of high multiplicity p? This is impossible even for $p = 2$ – it would contradict (1.7). However, this phenomenon can occur infinitely often.

There is a sequence $p_n \to \infty$, a function $f \in C(\mathbb{T})$ and a point c with the property that for infinitely many n, the best approximation T_n to f has a zero of multiplicity p_n at c. According to Zeller, this may be established as follows. Using the notation of Theorem 1.1, we put

$$f(t) = \sum_{k=1}^{\infty} a_k \cos^{q_k} N_k t \quad , \qquad \sum |a_k| < \infty \tag{1.9}$$

where q_k are odd positive integers which tend to infinity. The partial sum $S_k(t) := \sum_{i=1}^{k} a_i \cos^{q_i} N_i t$ is a polynomial of degree $N_k q_k$. If