ALM 25

Advanced Lectures in Mathematics

Handbook of Moduli

(Volume Ⅱ)

模手册(卷Ⅱ)

Editors: Gavril Farkas · Ian Morrison

高等教育出版社
HIGHER EDUCATION PRESS

ALM 25
Advanced Lectures in Mathematics

Handbook of Moduli

(Volume Ⅱ)

模手册（卷Ⅱ） Moshouce

Editors: Gavril Farkas · Ian Morrison

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

International Press

Higher Education Press Limited Company
4 Dewai Dajie, Beijing 100120, P. R. China, and
International Press
387 Somerville Ave, Somerville, MA, U. S. A.

The Handbook of Moduli was designed by Ian Morrison in LaTeX using a variant of the standard style files of the Higher Education Press. The text of the Handbook is set in ITC Giovanni and the mathematics in AMS Euler.

Giovanni was designed by Robert Slimbach in 1989 for ITC and was one of the early faces that earned him the Prix Charles Peignot, the Fields Medal of type design awarded "to a designer under the age of 35 who has made an outstanding contribution to type design". It combines the basic proportions of traditional oldstyle designs with the more even color and higher x-height of modern digital fonts to produce an inconspicuous but legible typeface.

Euler was designed in 1981 by Hermann Zapf, a major figure in 20th type design and a pioneer in digital typography, working in close cooperation with Donald Knuth, as an upright, cursive symbol font that would give the effect of mathematics handwritten on a blackboard. In 2008, Zapf reshaped many of the glyphs, with the assistance of Hans Hagen, Taco Hoekwater, and Volker RW Schaa, in order to harmonize the designs and bring them into line with contemporary standards of digital typography.

图书在版编目（CIP）数据

模手册.2 = Handbook of Moduli. Vol. Ⅱ：英文 /（德）法卡斯（Farkas, G.），（美）莫里森（Morrison, I.）编. —北京：高等教育出版社，2013.1
ISBN 978-7-04-035168-2

Ⅰ.①模… Ⅱ.①法… ②莫… Ⅲ.①代数几何－文集－英文 Ⅳ.①O187-53

中国版本图书馆CIP数据核字（2012）第242458号

策划编辑 王丽萍　　责任编辑 王丽萍　　封面设计 张申申　　责任印制 朱学忠

出版发行	高等教育出版社	咨询电话	400-810-0598
社　　址	北京市西城区德外大街4号	网　　址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印　　刷	涿州市星河印刷有限公司	网上订购	http://www.landraco.com
开　　本	787mm×1092mm 1/16		http://www.landraco.com.cn
印　　张	38	版　　次	2013年1月第1版
字　　数	720千字	印　　次	2013年1月第1次印刷
购书热线	010-58581118	定　　价	128.00元

物 料 号　35168-00

ADVANCED LECTURES IN MATHEMATICS

ADVANCED LECTURES IN MATHEMATICS

(*Executive Editors: Shing-Tung Yau, Kefeng Liu, Lizhen Ji*)

1. Superstring Theory (2007)
(Editors: Shing-Tung Yau, Kefeng Liu, Chongyuan Zhu)

2. Asymptotic Theory in Probability and Statistics with Applications (2007)
(Editors: Tze Leung Lai, Lianfen Qian, Qi-Man Shao)

3. Computational Conformal Geometry (2007)
(Authors: Xianfeng David Gu, Shing-Tung Yau)

4. Variational Principles for Discrete Surfaces (2007)
(Authors: Feng Luo, Xianfeng David Gu, Junfei Dai)

5. Proceedings of The 4th International Congress of Chinese Mathematicians Vol. I, II (2007)
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)

6. Geometry, Analysis and Topology of Discrete Groups (2008)
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)

7. Handbook of Geometric Analysis Vol. I (2008)
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)

8. Recent Developments in Algebra and Related Areas (2009)
(Editors: Chongying Dong, Fu-An Li)

9. Automorphic Forms and the Langlands Program (2009)
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)

10. Trends in Partial Differential Equations (2009)
(Editors: Baojun Bian, Shenghong Li, Xu-Jia Wang)

11. Recent Advances in Geometric Analysis (2009)
(Editors: Yng-Ing Lee, Chang-Shou Lin, Mao-Pei Tsui)

12. Cohomology of Groups and Algebraic K-theory (2009)
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)

13-14. Handbook of Geometric Analysis Vol. I, II (2010)
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)

15. An Introduction to Groups and Lattices (2010)
(Author: Robert L. Griess, Jr.)

16. Transformation Groups and Moduli Spaces of Curves (2010)
(Editors: Lizhen Ji, Shing-Tung Yau)

17-18. Geometry and Analysis Vol. I, II (2010)
(Editor: Lizhen Ji)

19. Arithmetic Geometry and Automorphic Forms (2011)
(Editors: James Cogdell, Jens Funke, Michael Rapoport, Tonghai Yang)

20. Surveys in Geometric Analysis and Relativity (2011)
(Editors: Hubert L. Bray, William P. Minicozzi II)

21. Advances in Geometric Analysis (2011)
(Editors: Stanisław Janeczko, Jun Li, Duong H. Phong)

22. Differential Geometry (2012)
(Editors: Yibing Shen, Zhongmin Shen, Shing-Tung Yau)

23. Recent Development in Geometry and Analysis (2012)
(Editors: Yuxin Dong, Jixiang Fu, Guozhen Lu, Weimin Sheng, Xiaohua Zhu)

24-26. Handbook of Moduli Vol. I, II, III (2012)
(Editors: Gavril Farkas, Ian Morrison)

The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

Contents

Volume I

Volume II

Volume III

Parameter spaces of curves

Joe Harris

Abstract. In this article I will try to survey the state of our knowledge (and the much greater area of our ignorance) of the geometry of spaces parametrizing curves in projective space.

Contents

1. Introduction

Robin Hartshorne, in [6], describes the problem of classifying algebraic varieties as the guiding problem of algebraic geometry. I'd agree, for the most part; and, since you're currently reading a book entitled "Handbook of Moduli," presumably you would too.

But the question remains: what exactly are we classifying? To be specific, consider the problem of smooth, complete algebraic curves over $\mathbb{C}$. If you ask mathematicians today to describe the problem of classifying curves, they would naturally take "curve" to mean "abstract curve," in which case the answer to the problem, "classify all smooth complete curves" would consist of two parts. Algebraic

2000 *Mathematics Subject Classification.* Primary 14Dxx; Secondary 14Dxx.
Key words and phrases. moduli.

curves are classified first by their sole discrete numerical invariant, the genus, which can assume any value $g \in \mathbb{N}$; and the set of curves of a given genus g has naturally the structure of an irreducible quasi-projective variety M_g. Beyond this, the problem of classifying algebraic curves consists of studying the geometry of the variety M_g, and of relating properties of curves to the loci in M_g of curves with that property.

If you had posed the same question to an algebraic geometer of the 19^{th} century, however, it would of necessity have been interpreted differently. Abstract curves didn't exist then (or, depending on your philosophical point of view, they hadn't been discovered); the word "curve" would have been taken to mean a subset of projective space defined by polynomial equations, smooth and irreducible of dimension 1. As such, a curve had not one but three numerical invariants: its degree d; the dimension r of the projective space in which it lay (or, more properly, the dimension of its span); and of course its genus. The problem of classifying all algebraic curves would thus amount to two things:

(1) To say for which triples g, r and d there exists a smooth, irreducible and nondegenerate curve of degree d and genus g in $\mathbb{P}^r$; and
(2) To describe, for each such triple (g, r, d), the geometry of the space $\mathcal{H}^\circ_{g,r,d}$ parametrizing such curves: its irreducible components, their dimensions and so on.

In this volume, there are many articles that address aspects of the problem of classification in its modern sense. But the classical version is still very much of interest, and has many fascinating aspects that are not fully understood: we haven't answered the first of the questions above; and we know the answer to the second only in an extremely limited range of cases. The goal of this article is to give a survey of what we do know about this problem, and likewise to suggest some of the numerous open problems.

The remainder of this paper will consist of three parts. In Section 2, we'll discuss the notion of parameter spaces of curves, and compare the two most commonly used such spaces, primarily the Hilbert scheme and the Kontsevich space. This may in a sense not be necessary if we're only concerned with smooth, irreducible curves in projective space, since the Hilbert scheme and the Kontsevich space have a common open subset parametrizing such curves (and indeed the reader can skip this section and go directly to the following ones). But for many purposes it's useful to have a compactification of the space of curves, and here the Hilbert scheme and the Kontsevich space differ dramatically, as we'll see.

In Section 3, we'll describe the conjectured answer to the Existence Problem, the first of the two questions listed above. This actually tells us a lot about curves of high genus: when g is more than roughly half the maximal possible genus of an irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$, in addition to simply saying which triples (g, r, d) occur, we learn about the geometry of such curves, and the dimension and irreducible components of their families. But for g below this bound, all we can say is that such curves exist; we can't say much about the spaces parametrizing them

Finally, in Section 4, we address this issue. We can in fact give a pretty explicit description of the spaces of curves of low genus, using what we know about the moduli space of abstract curves and Brill-Noether theory. Again, our knowledge—even conjectured—drops off as we approach the middle range of possible genera; we'll try to indicate what are some of the main unresolved questions in this area.

2. Parameter spaces

First of all, some terminology. We propose to call a space whose points correspond naturally to isomorphism classes of varieties or schemes X of a given type a *moduli space*; we'll call a space whose points correspond naturally to subschemes $X \subset Z$ of a fixed scheme Z (not up to isomorphism) a *parameter space*. There is not always a clear line dividing the two—for example, the Kontsevich space parameterizing stable maps has elements of both—but it does reflect an important duality in how we view geometric objects. One of the fundamental ideas underlying much recent progress in the theory of curves, for example, is the fact that whenever we have a one-parameter family $\{C_t \subset \mathbb{P}^r\}_{t\in\Delta}$ of curves in projective space, with C_t smooth for $t \neq t_0$, we have two distinct notions of the "limit" $\lim_{t\to t_0} C_t$ of the curves C_t: the *flat limit*, which is a subscheme of $\mathbb{P}^r$ whose geometry can be pretty much arbitrarily messy; and the *stable limit*, which is the limit of the abstract curves C_t and has at worst nodes as singularities. (Other articles in this volume discuss alternative notions of stability, and correspondingly alternative definitions of the limit of the abstract curves C_t; as for the flat limit, we really don't have much of an alternative to that.)

That said, what should we take as the parameter space for curves of degree d and genus g in $\mathbb{P}^r$? There are principally three answers to this question: the *Chow variety*, the *Hilbert scheme* and the *Kontsevich space*. These agree on the common open subset $\mathcal{H}^\circ_{g,r,d}$ parametrizing smooth curves (at least if we ignore the scheme structure on these spaces), but give very different compactifications of $\mathcal{H}^\circ_{g,r,d}$. Now, the questions we raised earlier—when do there exist such curves $C \subset \mathbb{P}^r$, what are the irreducible components of $\mathcal{H}^\circ_{g,r,d}$ and what are their dimensions—really don't depend on the choice of compactification, as long as we restrict our attention to the closure of $\mathcal{H}^\circ_{g,r,d}$ in each. But for many other questions it is important to have a complete parameter space, and so we start with a brief discussion of the properties of each. Actually, we'll pretty much ignore the Chow variety—in many ways, it has all the drawbacks of the Hilbert scheme and the Kontsevich space, and none of the virtues—and focus primarily on the other two.

The following discussion is adapted from a forthcoming book, *3264 and All That: Intersection Theory in Algebraic Geometry*, by the author and David Eisenbud.