

**ALM** 25

Advanced Lectures in Mathematics

# Handbook of Moduli

(Volume II)

模手册 (卷 II)

Editors: Gavril Farkas · Ian Morrison



高等教育出版社  
HIGHER EDUCATION PRESS

**ALM 25**

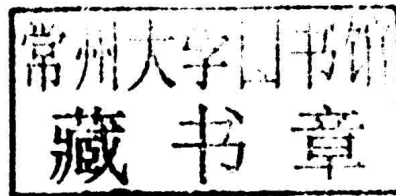
Advanced Lectures in Mathematics

# Handbook of Moduli

(Volume II)

模手册 (卷 II) Moshouce

Editors: Gavril Farkas · Ian Morrison



Copyright © 2013 by  
**Higher Education Press Limited Company**  
4 Dewai Dajie, Beijing 100120, P. R. China, and  
**International Press**  
387 Somerville Ave, Somerville, MA, U. S. A.

*All rights reserved. No part of this book may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without permission.*

The Handbook of Moduli was designed by Ian Morrison in  $\text{\LaTeX}$  using a variant of the standard style files of the Higher Education Press. The text of the Handbook is set in ITC Giovanni and the mathematics in AMS Euler.

Giovanni was designed by Robert Slimbach in 1989 for ITC and was one of the early faces that earned him the Prix Charles Peignot, the Fields Medal of type design awarded “to a designer under the age of 35 who has made an outstanding contribution to type design”. It combines the basic proportions of traditional oldstyle designs with the more even color and higher x-height of modern digital fonts to produce an inconspicuous but legible typeface.

Euler was designed in 1981 by Hermann Zapf, a major figure in 20<sup>th</sup> type design and a pioneer in digital typography, working in close cooperation with Donald Knuth, as an upright, cursive symbol font that would give the effect of mathematics handwritten on a blackboard. In 2008, Zapf reshaped many of the glyphs, with the assistance of Hans Hagen, Taco Hoekwater, and Volker RW Schaa, in order to harmonize the designs and bring them into line with contemporary standards of digital typography.

© 2008-2012 by Gavril Farkas and Ian Morrison. All rights reserved.

## 图书在版编目(CIP)数据

模手册. 2 = Handbook of Moduli. Vol. II: 英文 /  
(德)法卡斯(Farkas, G.), (美)莫里森(Morrison, I.) 编.  
—北京: 高等教育出版社, 2013. 1  
ISBN 978-7-04-035168-2

I. ①模… II. ①法… ②莫… III. ①代数几何-文集  
-英文 IV. ①O187-53

中国版本图书馆CIP数据核字(2012)第242458号

策划编辑 王丽萍      责任编辑 王丽萍      封面设计 张申申      责任印制 朱学忠

---

出版发行	高等教育出版社	咨询电话	400-810-0598
社 址	北京市西城区德外大街4号	网 址	<a href="http://www.hep.edu.cn">http://www.hep.edu.cn</a>
邮政编码	100120		<a href="http://www.hep.com.cn">http://www.hep.com.cn</a>
印 刷	涿州市星河印刷有限公司	网上订购	<a href="http://www.landrace.com">http://www.landrace.com</a>
开 本	787mm×1092mm 1/16		<a href="http://www.landrace.com.cn">http://www.landrace.com.cn</a>
印 张	38	版 次	2013年1月第1版
字 数	720千字	印 次	2013年1月第1次印刷
购书热线	010-58581118	定 价	128.00元

---

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换

版权所有 侵权必究

物料号 35168-00

## ADVANCED LECTURES IN MATHEMATICS

# ADVANCED LECTURES IN MATHEMATICS

---

(Executive Editors: Shing-Tung Yau, Kefeng Liu, Lizhen Ji)

1. Superstring Theory (2007)  
(Editors: Shing-Tung Yau, Kefeng Liu, Chongyuan Zhu)
2. Asymptotic Theory in Probability and Statistics with Applications (2007)  
(Editors: Tze Leung Lai, Lianfen Qian, Qi-Man Shao)
3. Computational Conformal Geometry (2007)  
(Authors: Xianfeng David Gu, Shing-Tung Yau)
4. Variational Principles for Discrete Surfaces (2007)  
(Authors: Feng Luo, Xianfeng David Gu, Junfei Dai)
5. Proceedings of The 4th International Congress of Chinese Mathematicians Vol. I, II (2007)  
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)
6. Geometry, Analysis and Topology of Discrete Groups (2008)  
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)
7. Handbook of Geometric Analysis Vol. I (2008)  
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)
8. Recent Developments in Algebra and Related Areas (2009)  
(Editors: Chongying Dong, Fu-An Li)
9. Automorphic Forms and the Langlands Program (2009)  
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)
10. Trends in Partial Differential Equations (2009)  
(Editors: Baojun Bian, Shenghong Li, Xu-Jia Wang)
11. Recent Advances in Geometric Analysis (2009)  
(Editors: Yng-Ing Lee, Chang-Shou Lin, Mao-Pei Tsui)
12. Cohomology of Groups and Algebraic K-theory (2009)  
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)
- 13–14. Handbook of Geometric Analysis Vol. I, II (2010)  
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)
15. An Introduction to Groups and Lattices (2010)  
(Author: Robert L. Griess, Jr.)
16. Transformation Groups and Moduli Spaces of Curves (2010)  
(Editors: Lizhen Ji, Shing-Tung Yau)
- 17–18. Geometry and Analysis Vol. I, II (2010)  
(Editor: Lizhen Ji)
19. Arithmetic Geometry and Automorphic Forms (2011)  
(Editors: James Cogdell, Jens Funke, Michael Rapoport, Tonghai Yang)
20. Surveys in Geometric Analysis and Relativity (2011)  
(Editors: Hubert L. Bray, William P. Minicozzi II)
21. Advances in Geometric Analysis (2011)  
(Editors: Stanisław Janeczko, Jun Li, Duong H. Phong)
22. Differential Geometry (2012)  
(Editors: Yibing Shen, Zhongmin Shen, Shing-Tung Yau)
23. Recent Development in Geometry and Analysis (2012)  
(Editors: Yuxin Dong, Jixiang Fu, Guozhen Lu, Weimin Sheng, Xiaohua Zhu)
- 24–26. Handbook of Moduli Vol. I, II, III (2012)  
(Editors: Gavril Farkas, Ian Morrison)

## ADVANCED LECTURES IN MATHEMATICS

### EXECUTIVE EDITORS

Shing-Tung Yau  
Harvard University  
Cambridge, MA. USA

Kefeng Liu  
University of California, Los Angeles  
Los Angeles, CA. USA

Lizhen Ji  
University of Michigan  
Ann Arbor, MI. USA

### EXECUTIVE BOARD

Chongqing Cheng  
Nanjing University  
Nanjing, China

Tatsien Li  
Fudan University  
Shanghai, China

Zhong-Ci Shi  
Institute of Computational Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Zhiying Wen  
Tsinghua University  
Beijing, China

Zhouping Xin  
The Chinese University of Hong Kong  
Hong Kong, China

Lo Yang  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Weiping Zhang  
Nankai University  
Tianjin, China

Xiping Zhu  
Sun Yat-sen University  
Guangzhou, China

Xiangyu Zhou  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China





*The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.*



# Contents

## Volume I

Preface	
<i>Gavril Farkas and Ian Morrison</i> .....	v
Logarithmic geometry and moduli	
<i>Dan Abramovich, Qile Chen, Danny Gillam, Yuhao Huang, Martin Olsson, Matthew Satriano and Shenghao Sun</i> .....	1
Invariant Hilbert schemes	
<i>Michel Brion</i> .....	63
Algebraic and tropical curves: comparing their moduli spaces	
<i>Lucia Caporaso</i> .....	119
A superficial working guide to deformations and moduli	
<i>F. Catanese</i> .....	161
Moduli spaces of hyperbolic surfaces and their Weil–Petersson volumes	
<i>Norman Do</i> .....	217
Equivariant geometry and the cohomology of the moduli space of curves	
<i>Dan Edidin</i> .....	259
Tautological and non-tautological cohomology of the moduli space of curves	
<i>C. Faber and R. Pandharipande</i> .....	293
Alternate compactifications of moduli spaces of curves	
<i>Maksym Fedorchuk and David Ishii Smyth</i> .....	331
The cohomology of the moduli space of Abelian varieties	
<i>Gerard van der Geer</i> .....	415
Moduli of K3 surfaces and irreducible symplectic manifolds	
<i>V. Gritsenko, K. Hulek and G.K. Sankaran</i> .....	459
Normal functions and the geometry of moduli spaces of curves	
<i>Richard Hain</i> .....	527

## Volume II

Parameter spaces of curves <i>Joe Harris</i> .....	1
Global topology of the Hitchin system <i>Tamás Hausel</i> .....	29
Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks <i>Stefan Kebekus</i> .....	71
Contractible extremal rays on $\overline{M}_{0,n}$ <i>Seán Keel and James McKernan</i> .....	115
Moduli of varieties of general type <i>János Kollár</i> .....	131
Singularities of stable varieties <i>Sándor J Kovács</i> .....	159
Soliton equations and the Riemann-Schottky problem <i>I. Krichever and T. Shiota</i> .....	205
GIT and moduli with a twist <i>Radu Laza</i> .....	259
Good degenerations of moduli spaces <i>Jun Li</i> .....	299
Localization in Gromov-Witten theory and Orbifold Gromov-Witten theory <i>Chiu-Chu Melissa Liu</i> .....	353
From WZW models to modular functors <i>Eduard Looijenga</i> .....	427
Shimura varieties and moduli <i>J.S. Milne</i> .....	467
The Torelli locus and special subvarieties <i>Ben Moonen and Frans Oort</i> .....	549

### Volume III

Birational geometry for nilpotent orbits <i>Yoshinori Namikawa</i> .....	1
Cell decompositions of moduli space, lattice points and Hurwitz problems <i>Paul Norbury</i> .....	39
Moduli of abelian varieties in mixed and in positive characteristic <i>Frans Oort</i> .....	75
Local models of Shimura varieties, I. Geometry and combinatorics <i>Georgios Pappas, Michael Rapoport and Brian Smithling</i> .....	135
Generalized theta linear series on moduli spaces of vector bundles on curves <i>Mihnea Popa</i> .....	219
Computer aided unirationality proofs of moduli spaces <i>Frank-Olaf Schreyer</i> .....	257
Deformation theory from the point of view of fibered categories <i>Mattia Talpo and Angelo Vistoli</i> .....	281
Mumford's conjecture — a topological outlook <i>Ulrike Tillmann</i> .....	399
Rational parametrizations of moduli spaces of curves <i>Alessandro Verra</i> .....	431
Hodge loci <i>Claire Voisin</i> .....	507
Homological stability for mapping class groups of surfaces <i>Nathalie Wahl</i> .....	547



# Parameter spaces of curves

Joe Harris

**Abstract.** In this article I will try to survey the state of our knowledge (and the much greater area of our ignorance) of the geometry of spaces parametrizing curves in projective space.

## Contents

1	Introduction	1
2	Parameter spaces	3
	2.1 Hilbert schemes	4
	2.2 The Kontsevich space	5
	2.3 Caveat: should we restrict attention to smooth curves, or reduced ones?	9
3	The existence problem	9
	3.1 Curves of maximal genus	10
	3.2 Curves of high genus	13
	3.3 Non-smoothable nodal curves	17
4	Curves of low and intermediate genus	18
	4.1 Curves of low genus	19
	4.2 Curves of intermediate genus	23

## 1. Introduction

Robin Hartshorne, in [6], describes the problem of classifying algebraic varieties as the guiding problem of algebraic geometry. I'd agree, for the most part; and, since you're currently reading a book entitled "Handbook of Moduli," presumably you would too.

But the question remains: what exactly are we classifying? To be specific, consider the problem of smooth, complete algebraic curves over  $\mathbb{C}$ . If you ask mathematicians today to describe the problem of classifying curves, they would naturally take "curve" to mean "abstract curve," in which case the answer to the problem, "classify all smooth complete curves" would consist of two parts. Algebraic

---

2000 *Mathematics Subject Classification.* Primary 14Dxx; Secondary 14Dxx.

*Key words and phrases.* moduli.

curves are classified first by their sole discrete numerical invariant, the genus, which can assume any value  $g \in \mathbb{N}$ ; and the set of curves of a given genus  $g$  has naturally the structure of an irreducible quasi-projective variety  $M_g$ . Beyond this, the problem of classifying algebraic curves consists of studying the geometry of the variety  $M_g$ , and of relating properties of curves to the loci in  $M_g$  of curves with that property.

If you had posed the same question to an algebraic geometer of the 19<sup>th</sup> century, however, it would of necessity have been interpreted differently. Abstract curves didn't exist then (or, depending on your philosophical point of view, they hadn't been discovered); the word "curve" would have been taken to mean a subset of projective space defined by polynomial equations, smooth and irreducible of dimension 1. As such, a curve had not one but three numerical invariants: its degree  $d$ ; the dimension  $r$  of the projective space in which it lay (or, more properly, the dimension of its span); and of course its genus. The problem of classifying all algebraic curves would thus amount to two things:

- (1) To say for which triples  $g, r$  and  $d$  there exists a smooth, irreducible and nondegenerate curve of degree  $d$  and genus  $g$  in  $\mathbb{P}^r$ ; and
- (2) To describe, for each such triple  $(g, r, d)$ , the geometry of the space  $\mathcal{H}_{g,r,d}^\circ$  parametrizing such curves: its irreducible components, their dimensions and so on.

In this volume, there are many articles that address aspects of the problem of classification in its modern sense. But the classical version is still very much of interest, and has many fascinating aspects that are not fully understood: we haven't answered the first of the questions above; and we know the answer to the second only in an extremely limited range of cases. The goal of this article is to give a survey of what we do know about this problem, and likewise to suggest some of the numerous open problems.

The remainder of this paper will consist of three parts. In Section 2, we'll discuss the notion of parameter spaces of curves, and compare the two most commonly used such spaces, primarily the Hilbert scheme and the Kontsevich space. This may in a sense not be necessary if we're only concerned with smooth, irreducible curves in projective space, since the Hilbert scheme and the Kontsevich space have a common open subset parametrizing such curves (and indeed the reader can skip this section and go directly to the following ones). But for many purposes it's useful to have a compactification of the space of curves, and here the Hilbert scheme and the Kontsevich space differ dramatically, as we'll see.

In Section 3, we'll describe the conjectured answer to the Existence Problem, the first of the two questions listed above. This actually tells us a lot about curves of high genus: when  $g$  is more than roughly half the maximal possible genus of an irreducible, nondegenerate curve of degree  $d$  in  $\mathbb{P}^r$ , in addition to simply saying which triples  $(g, r, d)$  occur, we learn about the geometry of such curves, and the dimension



and irreducible components of their families. But for  $g$  below this bound, all we can say is that such curves exist; we can't say much about the spaces parametrizing them

Finally, in Section 4, we address this issue. We can in fact give a pretty explicit description of the spaces of curves of low genus, using what we know about the moduli space of abstract curves and Brill-Noether theory. Again, our knowledge—even conjectured—drops off as we approach the middle range of possible genera; we'll try to indicate what are some of the main unresolved questions in this area.

## 2. Parameter spaces

First of all, some terminology. We propose to call a space whose points correspond naturally to isomorphism classes of varieties or schemes  $X$  of a given type a *moduli space*; we'll call a space whose points correspond naturally to subschemes  $X \subset Z$  of a fixed scheme  $Z$  (not up to isomorphism) a *parameter space*. There is not always a clear line dividing the two—for example, the Kontsevich space parameterizing stable maps has elements of both—but it does reflect an important duality in how we view geometric objects. One of the fundamental ideas underlying much recent progress in the theory of curves, for example, is the fact that whenever we have a one-parameter family  $\{C_t \subset \mathbb{P}^r\}_{t \in \Delta}$  of curves in projective space, with  $C_t$  smooth for  $t \neq t_0$ , we have two distinct notions of the “limit”  $\lim_{t \rightarrow t_0} C_t$  of the curves  $C_t$ : the *flat limit*, which is a subscheme of  $\mathbb{P}^r$  whose geometry can be pretty much arbitrarily messy; and the *stable limit*, which is the limit of the abstract curves  $C_t$  and has at worst nodes as singularities. (Other articles in this volume discuss alternative notions of stability, and correspondingly alternative definitions of the limit of the abstract curves  $C_t$ ; as for the flat limit, we really don't have much of an alternative to that.)

That said, what should we take as the parameter space for curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^r$ ? There are principally three answers to this question: the *Chow variety*, the *Hilbert scheme* and the *Kontsevich space*. These agree on the common open subset  $\mathcal{H}_{g,r,d}^\circ$  parametrizing smooth curves (at least if we ignore the scheme structure on these spaces), but give very different compactifications of  $\mathcal{H}_{g,r,d}^\circ$ . Now, the questions we raised earlier—when do there exist such curves  $C \subset \mathbb{P}^r$ , what are the irreducible components of  $\mathcal{H}_{g,r,d}^\circ$  and what are their dimensions—really don't depend on the choice of compactification, as long as we restrict our attention to the closure of  $\mathcal{H}_{g,r,d}^\circ$  in each. But for many other questions it is important to have a complete parameter space, and so we start with a brief discussion of the properties of each. Actually, we'll pretty much ignore the Chow variety—in many ways, it has all the drawbacks of the Hilbert scheme and the Kontsevich space, and none of the virtues—and focus primarily on the other two.

The following discussion is adapted from a forthcoming book, *3264 and All That: Intersection Theory in Algebraic Geometry*, by the author and David Eisenbud.