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MODEL THEORY

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INTRODUCTION

Should I begin by defining 'model theory'? This might be unsafe – do the readers get their money back if the definition doesn't match the contents? But here is an attempt at a definition: Model theory is the study of the construction and classification of structures within specified classes of structures.

A 'specified class of structures' is any class of structures that a mathematician might choose to name. For example it might be the class of abelian groups, or of Banach algebras, or sets with groups which act on them primitively. Thirty or forty years ago the founding fathers of model theory were particularly interested in classes specified by some set of axioms in first-order predicate logic – this would include the abelian groups but not the Banach algebras or the primitive groups. Today we have more catholic tastes, though many of our techniques work best on the first-order axiomatisable classes. One result of this broadening is that model theorists are usually much less interested than they used to be in the syntactical niceties of formal languages – if you want to know about formal languages today, you should go first to a computer scientist.

'Construction' means building structures, or sometimes families of structures, which have some feature that interests us. For example we might look for a graph which has very many automorphisms, or a group in which many systems of equations are solvable; or we might want a family of boolean algebras which can't be embedded in each other. 'Classifying' a class of structures means grouping the structures into subclasses in a useful way, and then proving that every structure in the collection does belong in just one of the subclasses. An archetypal example from algebra is the classification of vector spaces over a fixed field: we classify them by showing that each vector space has a unique dimension which determines it up to isomorphism. Model theory studies construction and classification in a broad setting, with methods that can be applied to many different classes of structure.

To give this book a shape, and to make it different from books which other people might write, I chose to concentrate on construction rather than classification. But that was twelve years ago, when (encouraged by Paul Cohn) I first sat down to write this book. Since then spells as editor, deputy head, dean etc. have destroyed any schedule that I ever had for writing the book, and of course the subject has moved on. The result was that far too much material accumulated. Some of it I diverted into three other books. Slightly over one megabyte has been shunted off to a file named Reject – I say this to appease readers who are annoyed to find no mention of their favourite topics or papers. The rejected material covers model theory of fields, atomic compact structures and infinitary languages, among several other things. These are valuable topics, but I simply ran out of space. (Other topics are missing out of brute ignorance – for example constructive model theory.)

Of the three other books, one has already appeared under the title *Building models by games*. It was originally a part of the present Chapter 8 (and it included the material that Paul Cohn had asked to see). A second book provisionally has the title *Structure and classification*, and will include much more stability theory. But several authors have already been kind enough to refer to the present book for some items in stability theory, and so I have taken the subject far enough to include those items. The third book will develop quasivarieties and Horn theories. Since this material has become more important for specification languages and logic programming than it ever was for model theory, that book will be aimed more at computer scientists.

After all these adjustments, the present book still has the emphases that I originally intended, though there is a lot more in it than I planned. Nearly every chapter is designed around some model-theoretic method of construction. In Chapter 1 it is diagrams, Chapter 3 includes Skolem hulls, Chapter 5 discusses interpretations as a method of construction, Chapter 6 tackles elementary amalgamation, Chapter 7 discusses omitting types and the Fraïssé construction, Chapter 8 is about existential closure, Chapter 9 deals with products, Chapter 10 builds saturated structures by unions of chains, Chapter 11 is about the Ehrenfeucht–Mostowski construction. That leaves Chapters 2, 4 and 12: Chapter 2 covers essential background material on languages, while Chapters 4 and 12 contain some recent developments of a geometric kind which I included because they are beautiful, important or both.

In the fourth century BC there was a bizarre philosophical debate about whether thought goes in straight lines or circles. Aristotle very sensibly supported the straight line theory, because (he maintained) proofs are linear.

Plato said circles, for astrological reasons which I wouldn't even wish to understand. But writing this book has convinced me that, just this once, Plato was right. There is no way that one can sensibly cover all the material in the book so that the later bits follow from the earlier ones. Time and again the more recent or sophisticated research throws up new information about the basic concepts. The later sections of several chapters, particularly Chapters 4 and 5, contain recent results which depend on things in later chapters. I trust this will cause no trouble; there are plenty of signposts in the text.

It would be hopeless to try to acknowledge all the people who have contributed to this book; they run into hundreds. But I warmly thank the people who read through sections – either of the final book or of parts now discarded – and gave me comments on them. They include Richard Archer, John Baldwin, Andreas Baudisch, Oleg Belegradek, Jeremy Clark, Paul Eklof, David Evans, Rami Grossberg, Deirdre Haskell, Lefty Kreouzis, Dugald Macpherson, Anand Pillay, Bruno Poizat, Philipp Rothmaler, Simon Thomas. I also thank David Tranah of Cambridge University Press for his encouragement and patience.

I owe a particular debt to Ian Hodkinson. There is no chance whatever that this book is free from errors; but thanks to his eagle eye and sound judgement, the number of mistakes is less than half what it would otherwise have been. His comments have led to improvements on practically every page. I say no more about his generous efforts for fear of getting him into trouble with his present employers, who must surely regard this as time misspent.

Finally a dedication. If this book is a success, I dedicate it to my students and colleagues, past and present, in the field of logic. Many of them appear in the pages which follow; but of those who don't, let me mention here two thoughtful and generous souls, Geoffrey Kneebone and Chris Fernau, both now retired, who ran the logic group of London University at Bedford College when I first came to London. If the book is not a success, I dedicate it to the burglars in Boulder, Colorado, who broke into our house and stole a television, two typewriters, my wife Helen's engagement ring and several pieces of cheese, somewhere about a third of the way through Chapter 8.

Acknowledgements. The passage of Hugh MacDiarmid, *On a raised beach* at the head of Chapter 2 is reprinted by permission of Martin Brian & O'Keefe Ltd, Blackheath. The lines from Eugène Ionesco, *La Cantatrice chauve* at the head of Chapter 3 are reprinted by permission of Editions Gallimard, Paris. The radiolarian skeleton at the head of Chapter 12 is reprinted by permission of John Wiley & Sons, Inc., New York.

NOTE ON NOTATION

Some exercises are marked with an asterisk *. This means only that I regard them as not the main exercises; maybe they assume specialist background, or they are very difficult, or they are off centre.

I assume Zermelo–Fraenkel set theory, ZFC. In particular I always assume the axiom of choice (except where the axiom itself is under discussion). I never assume the continuum hypothesis, existence of uncountable inaccessibles etc., without being honest about it.

The notation $x \subseteq y$ means that x is a subset of y; $x \subset y$ means that x is a proper subset of y. I write dom(f), im(f) for the domain and image of a function f. 'Greater than' means greater than, never 'greater than or equal to'. $\mathcal{P}(x)$ is the power set of x.

Ordinals are von Neumann ordinals, i.e. the predecessors of an ordinal α are exactly the elements of α. I use symbols α, β, γ, δ, i, j etc. for ordinals; δ is usually a limit ordinal. A cardinal κ is the smallest ordinal of cardinality κ, and the infinite cardinals are listed as ω_0, ω_1 etc. I use symbols κ, λ, μ, ν for cardinals; they are not assumed to be infinite unless the context clearly requires it (though I have probably slipped on this point once or twice). Natural numbers m, n etc. are the same thing as finite cardinals.

'Countable' means of cardinality ω. An infinite cardinal λ is a **regular** cardinal if it can't be written as the sum of fewer than λ cardinals which are all smaller than λ; otherwise it is **singular**. Every infinite successor cardinal κ^+ is regular. The smallest singular cardinal is $\omega_\omega = \sum_{n<\omega} \omega_n$. The **cofinality** cf($\alpha$) of an ordinal α is the least ordinal β such that α has a cofinal subset of order-type β; it can be shown that this ordinal β is either finite or regular. If α and β are ordinals, $\alpha\beta$ is the ordinal product consisting of β copies of α laid end to end. If κ and λ are cardinals, $\kappa\lambda$ is the cardinal product. The context should always show which of these products is intended.

Some facts of cardinal arithmetic are assembled at the beginning of section 10.4.

Sequences are well-ordered (except for indiscernible sequences in Chapter 11, and it is explicit there what is happening). I use the notation $\bar{x}$, $\bar{a}$ etc. for sequences $(x_0, x_1, \ldots)$, $(a_0, a_1, \ldots)$ etc., but loosely: the nth term of a sequence $\bar{x}$ may be x_n or $x(n)$ or something else, depending on the context, and some sequences start at x_1. Sequences of finite length are called **tuples**. The terms of a sequence are sometimes called its **items**, to avoid the ambiguity in the term 'term'. A sequence is said to be **non-repeating** if no item occurs twice or more in it. If $\bar{a}$ is a sequence $(a_0, a_1, \ldots)$ and f is a map, then $f\bar{a}$ is $(fa_0, fa_1, \ldots)$. The length of a sequence σ is written $\mathrm{lh}(\sigma)$. If σ is a sequence of length m and $n \leqslant m$, then $\sigma|n$ is the initial segment consisting of the first n terms of σ. The set of sequences of length γ whose items all come from the set X is written ${}^{\gamma}X$. Thus ${}^{n}2$ is the set of ordered n-tuples of 0's and 1's; ${}^{<\gamma}X$ is $\bigcup_{\alpha<\gamma} {}^{\alpha}X$. I write η, ζ, θ etc. for linear orderings; η^* is the ordering η run backwards.

I don't distinguish systematically between tuples and strings. If $\bar{a}$ and $\bar{b}$ are strings, $\bar{a}^\frown\bar{b}$ is the concatenated string consisting of $\bar{a}$ followed by $\bar{b}$; but often for simplicity I write it $\bar{a}\bar{b}$. There is a clash between the usual notation of model theory and the usual notation of groups: in model theory xy is the string consisting of x followed by y, but in groups it is x times y. One has to live with this; but where there is any ambiguity I have used $x^\frown y$ for the concatenated string and $x \cdot y$ for the group product.

Model-theoretic notation is defined as and when we need it. The most basic items appear in Chapter 1 and the first five sections of Chapter 2.

'I' means I, 'we' means we.

CONTENTS

1

Naming of parts

> Every person had in the beginning one only proper name, except the savages of Mount Atlas in Barbary, which were reported to be both nameless and dreamless.
>
> *William Camden*

In this first chapter we meet the main subject-matter of model theory: structures.

Every mathematician handles structures of some kind – be they modules, groups, rings, fields, lattices, partial orderings, Banach algebras or whatever. This chapter will define basic notions like 'element', 'homomorphism', 'substructure', and the definitions are not meant to contain any surprises. The notion of a (Robinson) 'diagram' of a structure may look a little strange at first, but really it is nothing more than a generalisation of the multiplication table of a group.

Nevertheless there is something that the reader may find unsettling. Model theorists are forever talking about symbols, names and labels. A group theorist will happily write the same abelian group multiplicatively or additively, whichever is more convenient for the matter in hand. Not so the model theorist: for him or her the group with '·' is one structure and the group with '+' is a different structure. Change the name and you change the structure.

This must look like pedantry. Model theory is an offshoot of mathematical logic, and I can't deny that some distinguished logicians have been pedantic about symbols. Nevertheless there are several good reasons why model theorists take the view that they do. For the moment let me mention two.

In the first place, we shall often want to compare two structures and study the homomorphisms from one to the other. What is a homomorphism? In the particular case of groups, a homomorphism from group G to group H is a map that carries multiplication in G to multiplication in H. There is an obvious way to generalise this notion to arbitrary structures: a homomorphism from structure A to structure B is a map which carries each operation of A to *the operation with the same name in B*.

Secondly, we shall often set out to build a structure with certain properties. One of the maxims of model theory is this: *name the elements of your* *structure first, then decide how they should behave.* If the names are well chosen, they will serve both as a scaffolding for the construction, and as raw materials.

Aha – says the group theorist – I see you aren't really talking about *written* symbols at all. For the purposes you have described, you only need to have formal labels for some parts of your structures. It should be quite irrelevant what kinds of thing your labels are; you might even want to have uncountably many of them.

Quite right. In fact we shall follow the lead of A. I. Mal'tsev [1936] and put no restrictions at all on what can serve as a name. For example any ordinal can be a name, and any mathematical object can serve as a name of itself. The items called 'symbols' in this book need not be written down. They need not even be dreamed.

1.1 Structures

We begin with a definition of 'structure'. It would have been possible to set up the subject with a slicker definition – say by leaving out clauses (1.2) and (1.4) below. But a little extra generality at this stage will save us endless complications later on.

A **structure** A is an object with the following four ingredients.

(1.1) A set called the **domain** of A, written $\text{dom}(A)$ or $\text{dom}\, A$ (some people call it the **universe** or **carrier** of A). The elements of $\text{dom}(A)$ are called the **elements** of the structure A. The **cardinality** of A, in symbols $|A|$, is defined to be the cardinality $|\text{dom}\, A|$ of $\text{dom}(A)$.

(1.2) A set of elements of A called **constant elements**, each of which is named by one or more **constants**. If c is a constant, we write c^A for the constant element named by c.

(1.3) For each positive integer n, a set of n-ary relations on $\text{dom}(A)$ (i.e. subsets of $(\text{dom}\, A)^n$), each of which is named by one or more n-ary **relation symbols**. If R is a relation symbol, we write R^A for the relation named by R.

(1.4) For each positive integer n, a set of n-ary operations on $\text{dom}(A)$ (i.e maps from $(\text{dom}\, A)^n$ to $\text{dom}(A)$), each of which is named by one or more n-ary **function symbols**. If F is a function symbol, we write F^A for the function named by F.

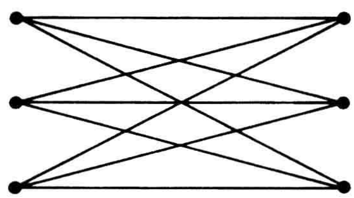
Except where we say otherwise, any of the sets (1.1)–(1.4) may be empty. As mentioned in the chapter introduction, the constant, relation and function 'symbols' can be any mathematical objects, not necessarily written symbols;

but for peace of mind one normally assumes that, for instance, a 3-ary relation symbol doesn't also appear as a 3-ary function symbol or a 2-ary relation symbol. We shall use capital letters $A, B, C, \ldots$ for structures.

Sequences of elements of a structure are written $\bar{a}$, $\bar{b}$ etc. A **tuple in** A (or **from** A) is a finite sequence of elements of A; it is an n-**tuple** if it has length n. Usually we leave it to the context to determine the length of a sequence or tuple.

This concludes the definition of 'structure'.

Example 1: ***Graphs.*** A **graph** consists of a set V (the set of **vertices**) and a set E (the set of **edges**), where each edge is a set of two distinct vertices. An edge $\{v, w\}$ is said to **join** the two vertices v and w. We can picture a finite graph by putting dots for the vertices and joining two vertices v, w by a line when $\{v, w\}$ is an edge:



One natural way to make a graph G into a structure is as follows. The elements of G are the vertices. There is one binary relation R^G; the ordered pair (v, w) lies in R^G if and only if there is an edge joining v to w.

Example 2: ***Linear orderings.*** Suppose $\leqslant$ linearly orders a set X. Then we can make $(X, \leqslant)$ into a structure A as follows. The domain of A is the set X. There is one binary relation symbol R, and its interpretation R^A is the ordering $\leqslant$. (In practice we would usually write the relation symbol as $\leqslant$ rather than R.)

Example 3: ***Groups.*** We can think of a group as a structure G with one constant 1 naming the identity 1^G, one binary function symbol $\cdot$ naming the group product operation $\cdot^G$, and one unary function symbol $^{-1}$ naming the inverse operation $^{(-1)^G}$. Another group H will have the same symbols 1, $\cdot$, $^{-1}$; then 1^H is the identity element of H, $\cdot^H$ is the product operation of H, and so on.

Example 4: ***Vector spaces.*** There are several ways to make a vector space into a structure, but here is the most convenient. Suppose V is a vector space over a field of scalars K. Take the domain of V to be the set of vectors of V. There is one constant element 0^V, the origin of the vector space. There is one binary operation, $+^V$, which is addition of vectors. There is a 1-ary operation $-^V$ for additive inverse; and for every scalar k there is a 1-ary operation k^V to represent multiplying a vector by k. Thus each scalar serves as a 1-ary function symbol. (In fact the symbol '$-$' is redundant, because $-^V$ is the same operation as $(-1)^V$.)

When we speak of vector spaces below, we shall assume that they are structures of this form (unless anything is said to the contrary). The same goes for modules, replacing the field K by a ring.

Two questions spring to mind. First, aren't these examples a little arbitrary? For example, why did we give the group structure a symbol for the multiplicative inverse $^{-1}$, but not a symbol for the commutator $[\,,]$? Why did we put into the linear ordering structure a symbol for the ordering $\leqslant$, but not one for the corresponding strict ordering $<$?

The answer is yes; these choices were arbitrary. But some choices are more sensible than others. We shall come back to this in the next section.

And second, *exactly* what is a structure? Our definition said nothing about the way in which the ingredients (1.1)–(1.4) are packed into a single entity.

True again. But this was a deliberate oversight – the packing arrangements will never matter to us. Some writers define A to be an ordered pair $\langle \mathrm{dom}(A), f\rangle$ where f is a function taking each symbol S to the corresponding item S^A. The important thing is to know what the symbols and the ingredients are, and this can be indicated in any reasonable way.

For example a model theorist may refer to the structure

$$\langle \mathbb{R}, +, -, \cdot, 0, 1, \leqslant\rangle.$$

With some common sense the reader can guess that this means the structure whose domain is the set of real numbers, with constants 0 and 1 naming the numbers 0 and 1, a 2-ary relation symbol $\leqslant$ naming the relation $\leqslant$, 2-ary function symbols $+$ and $\cdot$ naming addition and multiplication respectively, and a 1-ary function symbol naming minus.

Signatures

The **signature** of a structure A is specified by giving

(1.5) the set of constants of A, and for each separate $n > 0$, the set of n-ary relation symbols and the set of n-ary function symbols of A.

We shall assume that the signature of a structure can be read off uniquely from the structure.