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Theory of Multiobjective Optimization

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THEORY OF MULTIOBJECTIVE OPTIMIZATION

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THEORY OF MULTIOBJECTIVE OPTIMIZATION

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To our wives
Atsumi, Teruyo, and Yuko

PREFACE

This book presents in a comprehensive manner some salient theoretical aspects of multiobjective optimization. The authors had been involved in the special research project Environmental Science, sponsored by the Education Ministry of Japan, for more than a decade since 1970. Through the research activities, we became aware that an important thing is not merely to eliminate pollutants after they are discharged, but how to create a good environment from a holistic viewpoint. What, then, is a good environment? There are many factors: physical, chemical, biological, economic, social, and so on. In addition, to make the matter more difficult, there appear to be many conflicting values.

System scientific methodology seems effective for treating such a multiplicity of values. Its main concern is how to trade off these values. One of the major approaches is multiobjective optimization. Another is multiattribute utility analysis. The importance of these research themes has been widely recognized in theory and practice. Above all, the workshops at South Carolina in 1972 and at IIASA in 1975 have provided remarkable incentives to this field of research. Since then, much active research has been observed all over the world.

Although a number of books in this field have been published in recent years, they focus primarily on methodology. In spite of their importance, however, theoretical aspects of multiobjective optimization have never been dealt with in a unified way.

In Chapter 1 (Introduction), fundamental notions in multiobjective decision making and its historical background are briefly explained. Throughout this chapter, readers can grasp the purpose and scope of this volume.

Chapters 2–6 are the core of the book and are concerned with the mathematical theories in multiobjective optimization of existence, necessary and sufficient conditions of efficient solutions, characterization of efficient solutions, stability, and duality. Some of them are still developing, but we have tried to describe them in a unified way as much as possible.

Chapter 7 treats methodology including utility/value theory, stochastic dominance, and multiobjective programming methods. We emphasized critical points of these methods rather than a mere introduction. We hope that this approach will have a positive impact on future development of these areas.

The intended readers of this book are senior undergraduate students, graduate students, and specialists of decision making theory and mathematical programming, whose research fields are applied mathematics, electrical engineering, mechanical engineering, control engineering, economics, management sciences, operations research, and systems science. The book is self-contained so that it might be available either for reference and self-study or for use as a classroom text; only an elementary knowledge of linear algebra and mathematical programming is assumed.

Finally, we would like to note that we were motivated to write this book by a recommendation of the late Richard Bellman.

NOTATION AND SYMBOLS

$x := y$	x is defined as y
$x \in S$	x is a member of the set S
$x \notin S$	x is not a member of the set S
S^c	complement of the set S
cl S	closure of the set S
int S	interior of the set S
∂S	boundary of the set S
$S \subset T, T \supset S$	S is a subset of the set T
$S \cup T$	union of two sets S and T
$S \cap T$	intersection of two sets S and T
$S \backslash T$	difference between S and T, i.e., $S \cap T^c$
$S + T$	sum of two sets S and T, i.e., $S + T := \{s + t : s \in S$ and $t \in T\}$
$S \times T$	Cartesian product of the sets S and T, i.e., $S \times T := \{(s, t) : s \in S$ and $t \in T\}$
R^n	n-dimensional Euclidean space. A vector x in R^n is written $x = (x_1, x_2, \ldots, x_n)$, but when used in matrix calculations it is represented as a column vector, i.e., $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The corresponding row vector is $x^T = (x_1, x_2, \ldots, x_n)$. When $A = (a_{ij})$ is an $m \times n$ matrix with entry a_{ij} in its ith row and jth column, the

product Ax is the vector $y \in R^m$ with components $y_i = \sum_{j=1}^{m} a_{ij}x_j$, $i = 1, \ldots, m$.

R^p_+ — nonnegative orthant of R^p, i.e.,

$$R^p_+ := \{y \in R^p : y_i \geqq 0 \text{ for } i = 1, \ldots, p\}$$

$\mathring{R}^p_+$ — positive orthant of R^p, i.e., $\mathring{R}^p_+ := \{y \in R^p : y_i > 0 \text{ for } i = 1, \ldots, p\}$

$$S^p := \{y \in R^p_+ : \textstyle\sum_{i=1}^{p} y_i = 1\}$$
$$\mathring{S}^p := \{y \in \mathring{R}^p_+ : \textstyle\sum_{i=1}^{p} y_i = 1\}$$

For $x, y \in R^p$ and a pointed cone K of R^p with int $K \neq \varnothing$,

$x \geqq_K y$	is defined as	$x - y \in K$,
$x \geq_K y$	is defined as	$x - y \in K\backslash\{0\}$,
$x >_K y$	is defined as	$x - y \in \text{int } K$.

In particular, in case of $K = R^p_+$, the subscript K is omitted, namely,

$x \geqq y : x_i \geqq y_i$	for all	$i = 1, \ldots, p$;
$x \geq y : x \geqq y$	and	$x \neq y$;
$x > y : x_i > y_i$	for all	$i = 1, \ldots, p$.

$x \succ y$	x is preferred to y
$x \sim y$	x is indifferent to y
$\langle x, y \rangle$	inner product of the vectors x and y in the Euclidean space
$\|x\|$	Euclidean norm of the vector x in the Euclidean space
co S	convex hull of the set S
$T(S, y)$	tangent cone of the set S at y
$P(S)$	projecting cone of S
$0^+ Y$	recession cone of the set Y
Y^+	extended recession cone of the set Y
$\mathscr{E}(Y, D)$	efficient set of the set Y with respect to the domination structure (cone) D
$\mathscr{P}(Y, D)$	properly efficient set (in the sense of Benson) of the set Y with respect to the domination cone D
D-epi W	D-epigraph of the (point-to-set) map W

$\delta(\cdot \mid S)$	indicator function of the set S
$f'(x; d)$	one-sided directional derivative of the function f at x
$\nabla f(x)$	gradient of the function f at x, i.e., if $f: R^n \to R$, $$\nabla f(x) := \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$$
f^*	conjugate map (or function) of the function f
F^*	conjugate map of the point-to-set map F
$F^{**}(f^{**})$	biconjugate map of $F(f)$
$\partial f(x)$	subdifferential of the (vector-valued) function f at x
$\partial F(x; y)$	subdifferential of the point-to-set map F at $(x; y)$
$\mathrm{Max}_D\, Y$ ($\mathrm{Min}_D\, Y$)	set of the D-maximal (D-minimal) points of the set Y, i.e., $\mathrm{Max}_D\, Y := \mathscr{E}(Y, -D)$ ($\mathrm{Min}_D\, Y := \mathscr{E}(Y, D)$). In case of $D = R^p_+$, in particular, $\mathrm{Max}\, Y := \mathscr{E}(Y, -R^p_+)$ ($\mathrm{Min}\, Y := \mathscr{E}(Y, R^p_+)$)
w-$\mathrm{Max}_D\, Y$ (w-$\mathrm{Min}_D\, Y$)	set of the weak D-maximal (D-minimal) points of the set Y. In case of $D = R^p_+$, in particular, the subscript D is omitted.
w-Sup Y(w-Inf Y)	set of the weak supremal (infimum) points of the set Y
w-F^*	weak conjugate map of the point-to-set map F
w-F^{**}	weak biconjugate map of the point-to-set map F
w-$\partial F(x)$	weak subdifferential of the point-to-set map F at x
w-L	weak Lagrangian
max Y (**min** Y)	strong maximum (minimum) of the set Y
sup Y (**inf** Y)	strong supremum (infimum) of the set Y
$\mathbf{f}^*$	strong conjugate of the vector-valued function f
$\partial\mathbf{f}(x)$	strong subdifferential of the vector-valued function f at x

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5 LAGRANGE DUALITY

6 CONJUGATE DUALITY

7 METHODOLOGY

1 INTRODUCTION

Every day we encounter various kinds of decision making problems as managers, designers, administrative officers, mere individuals, and so on. In these problems, the final decision is usually made through several steps, even though they sometimes might not be perceived explicitly. Figure 1.1 shows a conceptual model of the decision making process. It implies that the final decision is made through three major models, the structure model, the impact model, and the evaluation model.

By structure modeling, we mean constructing a model in order to know the structure of the problem, what the problem is, which factors comprise the problem, how they interrelate, and so on. Through the process, the *objective* of the problem and *alternatives* to perform it are specified. Hereafter, we shall use the notation $\mathcal{O}$ for the objective and X for the set of alternatives, which is supposed to be a subset of an n-dimensional vector space. If we positively know a consequence caused by an alternative, the decision making is said to be *under certainty*; whereas if we cannot know a sure result because of some uncertain factor(s), the decision making is said to be *under uncertainty*. Furthermore, if we objectively or subjectively know the probability distribution of the possible consequences caused by an alternative, the decision making is said to be *under risk*. Even though the final objective might be a single entity, we encounter, in general, many subobjectives $\mathcal{O}_i$ on the way to the final objective. In this book, we shall consider decision making problems with multiple objectives. Interpretive structural modeling (ISM) (Warfield [W5]) can be applied effectively in order to obtain a hierarchical structure of the objectives.

In order to solve our decision making problem by some systems-analytical methods, we usually require that degrees of objectives be represented in numerical terms, which may be of multiple kinds even for one objective. In

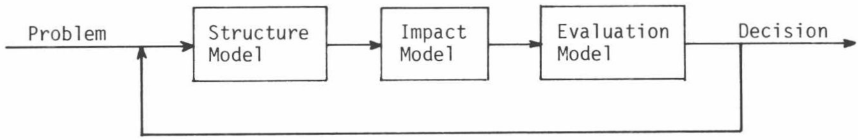

Fig. 1.1. Conceptual model of the decision making process.

order to exclude subjective value judgment at this stage, we restrict these numerical terms to physical measures (for example, money, weight, length, and time). As such a performance index, or criterion, for the objective $\mathcal{O}_i$, an *objective function* $f_i: X \to R^1$ is introduced, where R^1 denotes one-dimensional Euclidean space. The value $f_i(x)$ indicates how much impact is given on the objective $\mathcal{O}_i$ by performing an alternative x. Impact modeling is performed to identify these objective functions from various viewpoints such as physical, chemical, biological, social, economic, and so on. For convenience of mathematical treatment, we assume in this book that a smaller value for each objective function is preferred to a larger one. Now we can formulate our decision making problems as a *multiobjective optimization problem*:

$$\text{(P)} \qquad \text{Minimize} \qquad f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \qquad \text{over} \quad x \in X.$$

This kind of problem is also called a *vector optimization*. In some cases, some of the objective functions are required to be maintained under given levels prior to minimizing other objective functions. Denoting these objective functions by $g_j(x)$, we require that

$$g_j(x) \leq 0, \qquad j = 1, \ldots, m,$$

which, for convenience, is also supposed to represent some other technical constraints. Such a function $g_j(x)$ is generally called a *constraint function* in this book. According to the situation, we will consider either the problem (P) itself or (P) accompanied by the constraint $g_j(x) \leq 0$ $(j = 1, \ldots, m)$. Of course, an equality constraint $h_k(x) = 0$ can be embedded within two inequalities $h_k(x) \leq 0$ and $-h_k(x) \leq 0$, and, hence, it does not appear explicitly in this book.

Unlike traditional mathematical programming with a single objective function, an optimal solution in the sense of one that minimizes all the objective functions simultaneously does not necessarily exist in multiobjective optimization problems, and, hence, we are troubled with conflicts among objectives in decision making problems with multiple objectives. The final decision should be made by taking the total balance of objectives into account. Therefore, a new problem of value judgment called *value trade-off* arises. Evaluation modeling treats this problem that is peculiar to decision making with multiple objectives. Here we assume a *decision maker* who is responsible for the final decision. In some cases, there may be many decision makers, for which cases the decision making problems are called *group decision problems*. We will consider cases with a single decision maker in this book. The decision maker's value is usually represented by saying whether or not an alternative x is preferred to another alternative x', or equivalently whether or not $f(x)$ is preferred to $f(x')$. In other words, the decision maker's value is represented by some binary relation defined over X or $f(X)$. Since such a binary relation representing the decision maker's preference usually becomes an *order*, it is called a *preference order*. In this book, the decision maker's preference order is supposed to be defined on the so-called *criteria space* Y, which includes the set $f(X)$. Several kinds of preference orders will be possible, sometimes, the decision maker cannot judge whether or not $f(x)$ is preferred to $f(x')$. Roughly speaking, such an order that admits incomparability for a pair of objects is called a *partial order*, whereas the order requiring the comparability for every pair of objects is called a *weak order* (or *total order*). In practice, we often observe a partial order for the decision maker's preference. Unfortunately, however, an optimal solution in the sense of one that is most preferred with respect to the order, whence the notion of *optimality* does not necessarily exist for partial orders. Instead of strict optimality, we introduce in multiobjective optimization the notion of *efficiency*. A vector $f(\hat{x})$ is said to be efficient if there is no $f(x)$ $(x \in X)$ preferred to $f(\hat{x})$ with respect to the preference order. The final decision is usually made among the set of efficient solutions.

One approach to evaluation modeling is to find a scalar-valued function $u(f_1, \ldots, f_p)$ representing the decision maker's preference, which is called a *preference function* in this book. A preference function in decision making under risk is called a *utility function*, whereas the one in decision making under certainty is called a *value function*. The theory regarding existence, uniqueness, and practical representation of such a utility or value function is called the *utility and value theory*. Once we obtain such a preference function, our problem reduces to the traditional mathematical programming:

$$\text{Maximize} \qquad u(f_1(x), \ldots, f_p(x)) \qquad \text{over} \quad x \in X.$$

Another popular approach is the *interactive programming* that performs the solution search and evaluation modeling. In this approach, the solution is searched without identifying the preference function by eliciting iteratively some local information on the decision maker's preference.

Kuhn and Tucker [K10] first gave some interesting results concerning multiobjective optimization in 1951. Since then, research in this field has made remarkable progresss both theoretically and practically. Some of the earliest attempts to obtain conditions for efficiency were carried out by Kuhn and Tucker [K10], and Arrow *et al.* [A5]. Their research has been inherited by Da Cunha and Polak [D1], Neustadt [N14], Ritter [R4–R6], Smale [S10, S11], Aubin [A7], and others. After the 1970s, practical methodology such as utility and value analysis and interactive programming methods have been actively researched as tools for supporting decision making, and many books and conference proceedings on this topic have been published. (See, for example, Lee [L1], Cochrane and Zeleny [C12], Keeney and Raiffa [K6], Leitmann and Marzollo [L3], Leitmann [L2], Wilhelm [W15], Zeleny [Z4–Z6], Thiriez and Zionts [T14], Zionts [Z7], Starr and Zeleny [S13], Nijkamp and Delft [N18], Cohon [C13], Hwang and Masud [H17], Salkuvadze [S3], Fandel and Gal [F2], Rietveld [R3], Hwang and Yoon [H18], Morse [M5], Goicoeche *et al.* [G8], Hansen [H3], Chankong and Haimes [C6], and Grauer and Wierzbicki [G10].)

On the other hand, duality and stability, which play an important role in traditional mathematical programming, have been extended to multiobjective optimization since the late 1970s. Isermann [I5–I7] developed multiobjective duality in the linear case, while the results for nonlinear cases have been given by Schönfeld [S6], Rosinger [R10], Guglielmo [G12], Tanino and Sawaragi [T9, T11], Mazzoleni [M3], Bitran [B13], Brumelle [B21], Corley [C16], Jahn [J1], Kawasaki [K2, K3], Luc [L10], Nakayama [N5], and others. Stability for multiobjective optimization has been developed by Naccache [N2] and Tanino and Sawaragi [T10].

This book will be mainly concerned with some of the theoretical aspects in multiobjective optimization; in particular, we will focus on existence, necessary/sufficient conditions, stability, Lagrange duality, and conjugate duality for efficient solutions. In addition, some methodology such as utility and value theory and interactive programming methods will also be discussed.

Chapter 2 is devoted to some mathematical preliminaries. The first section gives a brief review of the elements of convex analysis that play an important role not only in traditional mathematical programming but also in multiobjective optimization. The second section describes point-to-set maps that play a very important role in the theory of multiobjective optimization, since the efficient solutions usually constitute a *set*. The concepts of continuity and convexity of point-to-set maps are introduced. These concepts are fundamental for existence and necessary/sufficient conditions for efficient solutions. The third section is concerned with a brief explanation of preference order and domination structures.

Chapter 3 begins with the introduction of several possible concepts for solutions in multiobjective optimization. Above all, efficient solutions will be the subject of primary consideration in subsequent theories. Next, some properties of efficient solutions, such as existence, external stability, connectedness, and necessary/sufficient conditions, will be discussed.