NONLINEAR
PHYSICAL
SCIENCE

Nail H. Ibragimov

Transformation Groups and Lie Algebras

变换群和李代数

高等教育出版社
HIGHER EDUCATION PRESS

Nail H. Ibragimov

Transformation Groups and Lie Algebras

变换群和李代数

BIANHUANQUN HE LIDAISHU

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

World Scientific

Author
Nail H. Ibragimov
Department of Mathematics and Science
Blekinge Institute of Technology
S-371 79 Karlskrona, Sweden

图书在版编目（CIP）数据

变换群和李代数 = Transformation Groups and Lie Algebras : 英文 /（瑞典）伊布拉基莫夫（Ibragimov, N.H.）著. — 北京 : 高等教育出版社，2013.3

（非线性物理科学 / 罗朝俊，（瑞典）伊布拉基莫夫主编）

ISBN 978-7-04-036741-6

Ⅰ. ①变… Ⅱ. ①伊… Ⅲ. ①变换群 - 英文②李代数 - 英文 Ⅳ. ① O152

中国版本图书馆 CIP 数据核字（2013）第 015562 号

策划编辑 王丽萍　　责任编辑 李 鹏　　封面设计 杨立新　　版式设计 王艳红
责任校对 杨凤玲　　责任印制 朱学忠

出版发行	高等教育出版社	咨询电话	400-810-0598
社　　址	北京市西城区德外大街 4 号	网　　址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印　　刷	涿州市星河印刷有限公司	网上订购	http://www.landraco.com
开　　本	787mm × 1092mm　1/16		http://www.landraco.com.cn
印　　张	12.25	版　　次	2013 年 3 月第 1 版
字　　数	210 千字	印　　次	2013 年 3 月第 1 次印刷
购书热线	010-58581118	定　　价	59.00 元

物 料 号　36741-00
Sales only inside the mainland of China
{仅限中国大陆地区销售}
本书海外版由 WSP 负责在中国大陆地区以外销售，ISBN 为 978-981-4460-84-2

NONLINEAR PHYSICAL SCIENCE

非线性物理科学

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

- New findings and discoveries in nonlinear physics and mathematics
- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

SERIES EDITORS

INTERNATIONAL ADVISORY BOARD

Preface

The term *transformation group* refers to the following properties of a collection G of invertible transformations $\bar{x} = T(x)$ of certain objects x :

1°. G contains the identity transformation I.
2°. G contains the inverse T^{-1} of any $T \in G$.
3°. G contains the product T_2T_1 of any $T_1, T_2 \in G$.

Note that the identity transformation I is defined by the equation $I(x) = x$. The product T_2T_1 is defined as a successive action of T_1 and T_2, i.e.

$$\big(T_2T_1\big)(x) = T_2\big(T_1(x)\big).$$

Finally, the inverse T^{-1} is defined by the equations $T^{-1}T = TT^{-1} = I$.

The group property of G is closely connected with the *invariance* of sets of the objects x under the transformations $T \in G$. We can formulate the statement in the following form.

Proposition. Let S be a set of objects x and G be the collection of all invertible transformations T defined on S and mapping any $x \in S$ into $T(x) = \bar{x} \in S$. Then G is a group.

Proof. Let us verify that the group properties $1° - 3°$ hold. The validity of the property 1° is obvious because $x \in S$ implies $I(x) = x \in S$. Hence, $I \in G$. Furthermore, $T(x) = \bar{x} \in S$ implies that $T^{-1}(\bar{x}) = x \in S$, and hence $T^{-1} \in G$, i.e. the property 2° is also satisfied. Finally, to verify the property 3°, we note that if $T_1, T_2 \in G$, then the action $T_2\big(T_1(x)\big)$ is defined because $T_1(x) \in S$, and $T_2\big(T_1(x)\big) \in S$ because T_2 maps any element of S into an element of S. Hence, $T_1, T_2 \in G$. This completes the proof.

In particular, if x denotes a solution of a given differential equation $F = 0$ and S is the totality of the solutions of $F = 0$, then the above statement shows that the collection of all transformations mapping any solution of $F = 0$ into a solution of the same differential equation compose a group. It is called the *group admitted by the differential equation,* or the *symmetry group* of the equation in question.

Part I of these notes introduces the reader to the basic concepts of the classical theory of local transformation groups and their Lie algebras. It has been designed for the graduate course on *Transformation groups and Lie algebras* that I have been teaching at Blekinge Institute of Technology, Karlskrona, Sweden, since 2002. The aim of this course was to augment a preliminary knowledge on symmetries of differential equations obtained by students during the course *Differential equations* based on my book [17], *A practical course in differential equations and mathematical modelling.*

Part II of these notes provides an easy to follow introduction to the new topic. It is based on my talks at various conferences, in particular on the plenary lecture at the International Workshop on "Differential equations and chaos" (University of Witwatersrand, Johannesburg, South Africa, January 1996). The final form of the presentation of this material, used in the present book, was prepared for my lectures "Approximate transformation groups" delivered for MSc students at Blekinge Institute of Technology since 2009.

Each part of the book contains an Assignment provided by detailed solutions of all problems. I hope that these assignments will be useful both for students and teachers.

Nail H. Ibragimov

Contents

Part I
Local Transformation Groups

Calculations show that groups admitted by differential equations involve one or more parameters and depend continuously on these parameters. This circumstance led Lie to the concept of *continuous transformation groups.* Multi-parameter continuous transformation groups are composed by *one-parameter groups* depending on a single continuous parameter. Each one-parameter group is determined by its *infinitesimal transformation* or the corresponding first-order linear differential operator termed the *generator* of the one-parameter group. One-parameter transformation groups and their generators are connected by means of the so-called *Lie equations.* Since the existence of solutions of the Lie equations is guaranteed, in general, only for values of the group parameter in a small neighborhood of its initial value, one arrives at what is called *local groups* of continuous transformations.

The generators of multi-parameter transformation groups form specific linear spaces known as *Lie algebras.* Description of continuous transformation groups in terms of their Lie algebras simplifies the calculation and use of groups admitted by differential equations significantly. Namely, the generators of continuous groups admitted by a given differential equation are defined by solving an over-determined system of linear differential equations known as *determining equations.* The characteristic property of determining equations is that *the totality of their solutions spans a Lie algebra.*

Due to the fundamental role of one-parameter groups in Lie's theory of continuous groups, it is natural to begin the study of the general theory of transformation groups and symmetries of differential equations by considering one-parameter groups and their generators.

Chapter 1
Preliminaries

This chapter introduces the reader to a general idea of transformations and exhibits a variety of transformation groups. The duality between changes of frames of reference and point transformations is useful in group analysis. We discuss the idea of the duality in this chapter and will employ it in the next chapter for the prolongation of point transformation groups to derivatives.

1.1 Changes of frames of reference and point transformations

1.1.1 Translations

Consider, in the (x,y) plane, a point P having the coordinates (x,y) in the rectangular Cartesian reference frame with the axes Ox, Oy. Let $\boldsymbol{e} = (e_1, e_2)$ be a fixed unit vector. Consider a new pair of rectangular axes $\overline{O}\bar{x}, \overline{O}\bar{y}$ parallel to the former axes such that $\overline{O}$ has the coordinates $(-ae_1, -be_2)$ with respect to the original frame of reference, where a is an arbitrary real parameter. Then the coordinates $(\bar{x}, \bar{y})$ of the point P in the new frame of reference are given by

$$\bar{x} = x + ae_1, \quad \bar{y} = y + be_2. \tag{1.1.1}$$

An alternative interpretation of Eqs. (1.1.1) is as follows. One ignores the new axes $\overline{O}\bar{x}, \overline{O}\bar{y}$ and regards (x,y) and $(\bar{x}, \bar{y})$ as the coordinates of points P and $\overline{P}$, respectively, each referred to the original frame Ox, Oy. Then Eqs. (1.1.1) define a transformation of the point $P(x,y)$ into the new position $\overline{P}(\bar{x}, \bar{y})$ in the (x,y) plane. Accordingly, equations (1.1.1) determine the displacement (translation) of all points P of the plane through the distance a in the direction of the vector $\boldsymbol{e}$.

1.1.2 Rotations

Consider again the rectangular Cartesian reference frame with the axes Ox, Oy. Let $O\bar{x}, O\bar{y}$ be the new pair of axes obtained by rotating the original axes round the origin

O counter-clockwise through an angle a. Let (x,y) and $(\bar{x},\bar{y})$ be the coordinates of a point P referred to the axes Ox, Oy and $O\bar{x}, O\bar{y}$, respectively. Then we have

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a. \tag{1.1.2}$$

Indeed, in the polar coordinates (r,θ), connected with the Cartesian coordinates by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1.1.3}$$

the rotation by the angle a about the origin clockwise is written

$$\bar{r} = r, \quad \bar{\theta} = \theta - a. \tag{1.1.4}$$

Equations (1.1.3), (1.1.4) yield the following transformation:

$$\bar{x} = \bar{r}\cos\bar{\theta} = r\cos(\theta - a), \quad \bar{y} = \bar{r}\sin\bar{\theta} = r\sin(\theta - a).$$

Expanding $\cos(\theta - a)$ and $\sin(\theta - a)$ and substituting $r\cos\theta = x,\ r\sin\theta = y$, one arrives at Eqs. (1.1.2).

An alternative interpretation of Eqs. (1.1.2) is as follows. We regard (x,y) and $(\bar{x},\bar{y})$ as the coordinates of the points P and $\overline{P}$, respectively, each referred to the same axes Ox, Oy. Then Eqs. (1.1.2) accomplish the rotation of all points of the plane about O clockwise through the angle a.

1.1.3 Galilean transformation

Everyone travelling by train can observe the duality between uniform motions of his local frame of reference (a train) and outside points (people or other objects on a depot). This remarkable exhibition of the duality, when one cannot determine who is actually moving, is known in the classical mechanics as *Galileo's relativity principle*. It is equivalent to the invariance of equations of motion of mechanical systems under the transformation

$$\bar{t} = t, \quad \overline{\boldsymbol{x}} = \boldsymbol{x} + t\boldsymbol{V}, \tag{1.1.5}$$

where $\boldsymbol{V}$ is the constant velocity. Differentiation of $\overline{\boldsymbol{x}}$ with respect to $\bar{t} = t$ yields

$$\overline{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{V}. \tag{1.1.6}$$

The transformation (1.1.6) of the velocity is a mathematical expression of Galileo's relativity principle. The transformation (1.1.5) is known as the Galilean transformation and lies at the core of the *Galilean group* which is one of the most important groups in non-relativistic physics.

1.2 Introduction of transformation groups

1.2.1 Definitions and examples

We will consider invertible transformations in an n-dimensional Euclidean space $\mathbb{R}^n$ defined, in coordinates, by equations of the form

$$\bar{x}^i = f^i(x), \quad i = 1,\ldots,n, \tag{1.2.1}$$

where the vector-function $f = (f^1,\ldots,f^n)$ is continuous together with its derivatives involved in further discussions. Since the transformation (1.2.1) is invertible, there exists the inverse transformation

$$x^i = (f^{-1})^i(\bar{x}), \quad i = 1,\ldots,n. \tag{1.2.2}$$

Let us denote the transformation (1.2.1) by T and its inverse (1.2.2) by T^{-1}. Thus, T carries any point

$$x = (x^1,\ldots,x^n) \in \mathbb{R}^n$$

into a new position

$$\bar{x} = (\bar{x}^1,\ldots,\bar{x}^n) \in \mathbb{R}^n,$$

and T^{-1} returns $\bar{x}$ into the original position x. It is assumed that the coordinates x^i and $\bar{x}^i$ of points x and $\bar{x}$, respectively, are referred to one and the same coordinate system. The identical transformation

$$\bar{x}^i = x^i, \quad i = 1,\ldots,n, \tag{1.2.3}$$

will be denoted by I.

Let T_1 and T_2 be two transformations of the form (1.2.1) with functions f_1^i and f_2^i, respectively. Their *product* T_2T_1 (termed also *composition* and denoted by $T_2 \circ T_1$) is defined as the consecutive application of these transformations and is given by

$$\bar{\bar{x}}^i = f_2^i(\bar{x}) = f_2^i(f_1(x)), \quad i = 1,\ldots,n. \tag{1.2.4}$$

The geometric interpretation of the product is as follows. Since T_1 carries the point x to the point $\bar{x} = T_1(x)$, which T_2 carries to the new position $\bar{\bar{x}} = T_2(\bar{x})$, the effect of the product T_2T_1 is to carry x directly to its final location $\bar{\bar{x}}$, without a stopover at $\bar{x}$. Thus, equation (1.2.4) means that

$$\bar{\bar{x}} \stackrel{\text{def}}{=} T_2(\bar{x}) = T_2T_1(x). \tag{1.2.5}$$

In this notation, the definition of the inverse transformation (1.2.2) means

$$TT^{-1} = T^{-1}T = I. \tag{1.2.6}$$

Definition 1.1. A set G of transformations (1.2.1) in $\mathbb{R}^n$ containing the identity I is called a transformation group if it contains the inverse T^{-1} of every transformation $T \in G$ and the product T_1T_2 of any transformations $T_1, T_2 \in G$. Thus, the attributes of the group G are:

$$I \in G, \quad \text{and} \quad T^{-1} \in G, \quad T_1T_2 \in G \quad \text{whenever } T, T_1, T_2 \in G. \tag{1.2.7}$$

Example 1.1. The set $G = \{I, T_1, \ldots, T_5\}$ of the transformations

$$\begin{aligned} &I : \bar{x} = x, \qquad T_1 : \bar{x} = 1 - x, \qquad T_2 : \bar{x} = \frac{1}{x}, \\ &T_3 : \bar{x} = \frac{1}{1-x}, \quad T_4 : \bar{x} = \frac{x}{x-1}, \quad T_5 : \bar{x} = \frac{x-1}{x} \end{aligned} \tag{1.2.8}$$

on the straight line is a group containing six elements (see, e.g., [6], §9). The group properties (1.2.7) can be verified by computing the inverses and products of the transformations (1.2.8), e.g.

$$\begin{aligned} &T_1^{-1} = T_1,\ T_2^{-1} = T_2,\ T_3^{-1} = T_5,\ T_4^{-1} = T_4,\ T_5^{-1} = T_3, \\ &T_1^2 = I, \quad T_2^2 = I, \quad T_3^2 = T_5, \quad T_4^2 = I, \quad T_5^2 = T_3, \\ &T_2T_1 = T_3, \quad T_1T_2 = T_5, \quad T_3T_1 = T_2, \quad T_1T_3 = T_4. \end{aligned} \tag{1.2.9}$$

Example 1.2. Consider the set G of all translations (displacements) T_a :

$$\bar{x} = x + a \tag{1.2.10}$$

on the straight line. Since $\bar{x} = x$ when $a = 0$, the set G contains the identity $I = T_0$. Furthermore, the combined effect of two translations, T_a and T_b, acting in succession, is to displace x through the distance $a + b$. Hence,

$$T_bT_a = T_{a+b}. \tag{1.2.11}$$

Equation (1.2.11) shows that

$$T_a^{-1} = T_{-a}.$$

Thus, the transformations (1.2.10) obey the group properties (1.2.7), and hence define a *one-parameter group* G, i.e. a group containing one arbitrary parameter a. This group is known as the *translation group* and provides one of the simplest illustrations to the following definition.

Definition 1.2. A set G of transformations T_a in $\mathbb{R}^n$ depending continuously on a parameter a, where a ranges over all real numbers from a given interval $U \subset R$, is called a *one-parameter group* if there is a unique value $a = a_0$ in U providing the identical transformation, $T_{a_0} = I$, and the following conditions hold for all $a, b \in U$:

$$T_a^{-1} = T_{a^{-1}} \in G, \quad T_bT_a = T_c \in G, \tag{1.2.12}$$