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Nail H. Ibragimov

Transformation Groups and Lie Algebras

变换群和李代数



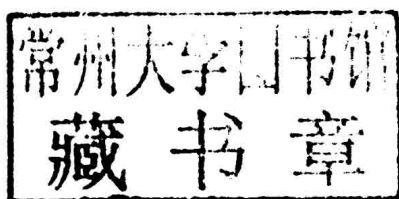
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NONLINEAR PHYSICAL SCIENCE

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Preface

The term *transformation group* refers to the following properties of a collection G of invertible transformations $\bar{x} = T(x)$ of certain objects x :

- 1°. G contains the identity transformation I .
- 2°. G contains the inverse T^{-1} of any $T \in G$.
- 3°. G contains the product $T_2 T_1$ of any $T_1, T_2 \in G$.

Note that the identity transformation I is defined by the equation $I(x) = x$. The product $T_2 T_1$ is defined as a successive action of T_1 and T_2 , i.e.

$$(T_2 T_1)(x) = T_2(T_1(x)).$$

Finally, the inverse T^{-1} is defined by the equations $T^{-1}T = TT^{-1} = I$.

The group property of G is closely connected with the *invariance* of sets of the objects x under the transformations $T \in G$. We can formulate the statement in the following form.

Proposition. Let S be a set of objects x and G be the collection of all invertible transformations T defined on S and mapping any $x \in S$ into $T(x) = \bar{x} \in S$. Then G is a group.

Proof. Let us verify that the group properties 1° – 3° hold. The validity of the property 1° is obvious because $x \in S$ implies $I(x) = x \in S$. Hence, $I \in G$. Furthermore, $T(x) = \bar{x} \in S$ implies that $T^{-1}(\bar{x}) = x \in S$, and hence $T^{-1} \in G$, i.e. the property 2° is also satisfied. Finally, to verify the property 3°, we note that if $T_1, T_2 \in G$, then the action $T_2(T_1(x))$ is defined because $T_1(x) \in S$, and $T_2(T_1(x)) \in S$ because T_2 maps any element of S into an element of S . Hence, $T_1, T_2 \in G$. This completes the proof.

In particular, if x denotes a solution of a given differential equation $F = 0$ and S is the totality of the solutions of $F = 0$, then the above statement shows that the collection of all transformations mapping any solution of $F = 0$ into a solution of the same differential equation compose a group. It is called the *group admitted by the differential equation*, or the *symmetry group* of the equation in question.

Part I of these notes introduces the reader to the basic concepts of the classical theory of local transformation groups and their Lie algebras. It has been designed for the graduate course on *Transformation groups and Lie algebras* that I have been teaching at Blekinge Institute of Technology, Karlskrona, Sweden, since 2002. The

aim of this course was to augment a preliminary knowledge on symmetries of differential equations obtained by students during the course *Differential equations* based on my book [17], *A practical course in differential equations and mathematical modelling*.

Part II of these notes provides an easy to follow introduction to the new topic. It is based on my talks at various conferences, in particular on the plenary lecture at the International Workshop on “Differential equations and chaos” (University of Witwatersrand, Johannesburg, South Africa, January 1996). The final form of the presentation of this material, used in the present book, was prepared for my lectures “Approximate transformation groups” delivered for MSc students at Blekinge Institute of Technology since 2009.

Each part of the book contains an Assignment provided by detailed solutions of all problems. I hope that these assignments will be useful both for students and teachers.

Nail H. Ibragimov

Contents

Preface	v
Part I Local Transformation Groups	1
1 Preliminaries	3
1.1 Changes of frames of reference and point transformations	3
1.1.1 Translations	3
1.1.2 Rotations	3
1.1.3 Galilean transformation	4
1.2 Introduction of transformation groups	5
1.2.1 Definitions and examples	5
1.2.2 Different types of groups	10
1.3 Some useful groups	13
1.3.1 Finite continuous groups on the straight line	13
1.3.2 Groups on the plane	14
1.3.3 Groups in \mathbb{R}^n	19
Exercises to Chapter 1	21
2 One-parameter groups and their invariants	23
2.1 Local groups of transformations	23
2.1.1 Notation and definition	23
2.1.2 Groups written in a canonical parameter	25
2.1.3 Infinitesimal transformations and generators	25
2.1.4 Lie equations	27
2.1.5 Exponential map	29
2.1.6 Determination of a canonical parameter	32
2.2 Invariants	34
2.2.1 Definition and infinitesimal test	34
2.2.2 Canonical variables	36
2.2.3 Construction of groups using canonical variables	38

2.2.4	Frequently used groups in the plane	40
2.3	Invariant equations	41
2.3.1	Definition and infinitesimal test	41
2.3.2	Invariant representation of invariant manifolds	43
2.3.3	Proof of Theorem 2.9	44
2.3.4	Examples on Theorem 2.9	45
	Exercises to Chapter 2	47
3	Groups admitted by differential equations	51
3.1	Preliminaries	51
3.1.1	Differential variables and functions	51
3.1.2	Point transformations	53
3.1.3	Frame of differential equations	53
3.2	Prolongation of group transformations	54
3.2.1	One-dimensional case	54
3.2.2	Prolongation with several differential variables	55
3.2.3	General case	56
3.3	Prolongation of group generators	56
3.3.1	One-dimensional case	56
3.3.2	Several differential variables	59
3.3.3	General case	60
3.4	First definition of symmetry groups	62
3.4.1	Definition	62
3.4.2	Examples	62
3.5	Second definition of symmetry groups	67
3.5.1	Definition and determining equations	67
3.5.2	Determining equation for second-order ODEs	68
3.5.3	Examples on solution of determining equations	68
	Exercises to Chapter 3	73
4	Lie algebras of operators	75
4.1	Basic definitions	75
4.1.1	Commutator	75
4.1.2	Properties of the commutator	77
4.1.3	Properties of determining equations	79
4.1.4	Lie algebras	80
4.2	Basic properties	81
4.2.1	Notation	81
4.2.2	Subalgebra and ideal	81
4.2.3	Derived algebras	82
4.2.4	Solvable Lie algebras	83
4.3	Isomorphism and similarity	84
4.3.1	Isomorphic Lie algebras	84
4.3.2	Similar Lie algebras	86
4.4	Low-dimensional Lie algebras	88

4.4.1	One-dimensional algebras	88
4.4.2	Two-dimensional algebras in the plane	89
4.4.3	Three-dimensional algebras in the plane	97
4.4.4	Three-dimensional algebras in \mathbb{R}^3	99
4.5	Lie algebras and multi-parameter groups	101
4.5.1	Definition of multi-parameter groups	101
4.5.2	Construction of multi-parameter groups	102
	Exercises to Chapter 4	104
5	Galois groups via symmetries	107
5.1	Preliminaries	107
5.2	Symmetries of algebraic equations	108
5.2.1	Determining equation	108
5.2.2	First example	109
5.2.3	Second example	111
5.2.4	Third example	112
5.3	Construction of Galois groups	113
5.3.1	First example	113
5.3.2	Second example	114
5.3.3	Third example	115
5.3.4	Concluding remarks	116
	Assignment to Part I	117
	Part II Approximate Transformation Groups	125
6	Preliminaries	127
6.1	Motivation	127
6.2	A sketch on Lie transformation groups	129
6.2.1	One-parameter transformation groups	129
6.2.2	Canonical parameter	130
6.2.3	Group generator and Lie equations	131
6.2.4	Exponential map	133
6.3	Approximate Cauchy problem	134
6.3.1	Notation	134
6.3.2	Definition of the approximate Cauchy problem	136
7	Approximate transformations	139
7.1	Approximate transformations defined	139
7.2	Approximate one-parameter groups	140
7.2.1	Introductory remark	140
7.2.2	Definition of one-parameter approximate transformation groups	140
7.2.3	Generator of approximate transformation group	141
7.3	Infinitesimal description	142
7.3.1	Approximate Lie equations	142

7.3.2	Approximate exponential map	146
	Exercises to Chapter 7	150
8	Approximate symmetries	151
8.1	Definition of approximate symmetries	151
8.2	Calculation of approximate symmetries	152
8.2.1	Determining equations	152
8.2.2	Stable symmetries	152
8.2.3	Algorithm for calculation	153
8.3	Examples	154
8.3.1	First example	154
8.3.2	Approximate commutator and Lie algebras	155
8.3.3	Second example	156
8.3.4	Third example	157
	Exercises to Chapter 8	158
9	Applications	161
9.1	Integration of equations with a small parameter using approximate symmetries	161
9.1.1	Equation having no exact point symmetries	161
9.1.2	Utilization of stable symmetries	162
9.2	Approximately invariant solutions	166
9.2.1	Nonlinear wave equation	166
9.2.2	Approximate travelling waves of KdV equation	170
9.3	Approximate conservation laws	172
	Exercises to Chapter 9	174
	Assignment to Part II	175
	Bibliography	181
	Index	183

Part I
Local Transformation Groups

Calculations show that groups admitted by differential equations involve one or more parameters and depend continuously on these parameters. This circumstance led Lie to the concept of *continuous transformation groups*. Multi-parameter continuous transformation groups are composed by *one-parameter groups* depending on a single continuous parameter. Each one-parameter group is determined by its *infinitesimal transformation* or the corresponding first-order linear differential operator termed the *generator* of the one-parameter group. One-parameter transformation groups and their generators are connected by means of the so-called *Lie equations*. Since the existence of solutions of the Lie equations is guaranteed, in general, only for values of the group parameter in a small neighborhood of its initial value, one arrives at what is called *local groups* of continuous transformations.

The generators of multi-parameter transformation groups form specific linear spaces known as *Lie algebras*. Description of continuous transformation groups in terms of their Lie algebras simplifies the calculation and use of groups admitted by differential equations significantly. Namely, the generators of continuous groups admitted by a given differential equation are defined by solving an over-determined system of linear differential equations known as *determining equations*. The characteristic property of determining equations is that *the totality of their solutions spans a Lie algebra*.

Due to the fundamental role of one-parameter groups in Lie's theory of continuous groups, it is natural to begin the study of the general theory of transformation groups and symmetries of differential equations by considering one-parameter groups and their generators.

Chapter 1

Preliminaries

This chapter introduces the reader to a general idea of transformations and exhibits a variety of transformation groups. The duality between changes of frames of reference and point transformations is useful in group analysis. We discuss the idea of the duality in this chapter and will employ it in the next chapter for the prolongation of point transformation groups to derivatives.

1.1 Changes of frames of reference and point transformations

1.1.1 Translations

Consider, in the (x, y) plane, a point P having the coordinates (x, y) in the rectangular Cartesian reference frame with the axes Ox, Oy . Let $e = (e_1, e_2)$ be a fixed unit vector. Consider a new pair of rectangular axes $\overline{Ox}, \overline{Oy}$ parallel to the former axes such that \overline{O} has the coordinates $(-ae_1, -be_2)$ with respect to the original frame of reference, where a is an arbitrary real parameter. Then the coordinates (\bar{x}, \bar{y}) of the point P in the new frame of reference are given by

$$\bar{x} = x + ae_1, \quad \bar{y} = y + be_2. \quad (1.1.1)$$

An alternative interpretation of Eqs. (1.1.1) is as follows. One ignores the new axes $\overline{Ox}, \overline{Oy}$ and regards (x, y) and (\bar{x}, \bar{y}) as the coordinates of points P and \bar{P} , respectively, each referred to the original frame Ox, Oy . Then Eqs. (1.1.1) define a transformation of the point $P(x, y)$ into the new position $\bar{P}(\bar{x}, \bar{y})$ in the (x, y) plane. Accordingly, equations (1.1.1) determine the displacement (translation) of all points P of the plane through the distance a in the direction of the vector e .

1.1.2 Rotations

Consider again the rectangular Cartesian reference frame with the axes Ox, Oy . Let $\overline{Ox}, \overline{Oy}$ be the new pair of axes obtained by rotating the original axes round the origin

O counter-clockwise through an angle a . Let (x, y) and (\bar{x}, \bar{y}) be the coordinates of a point P referred to the axes Ox, Oy and $O\bar{x}, O\bar{y}$, respectively. Then we have

$$\bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a. \quad (1.1.2)$$

Indeed, in the polar coordinates (r, θ) , connected with the Cartesian coordinates by the equations

$$x = r \cos \theta, \quad y = r \sin \theta, \quad (1.1.3)$$

the rotation by the angle a about the origin clockwise is written

$$\bar{r} = r, \quad \bar{\theta} = \theta - a. \quad (1.1.4)$$

Equations (1.1.3), (1.1.4) yield the following transformation:

$$\bar{x} = \bar{r} \cos \bar{\theta} = r \cos(\theta - a), \quad \bar{y} = \bar{r} \sin \bar{\theta} = r \sin(\theta - a).$$

Expanding $\cos(\theta - a)$ and $\sin(\theta - a)$ and substituting $r \cos \theta = x$, $r \sin \theta = y$, one arrives at Eqs. (1.1.2).

An alternative interpretation of Eqs. (1.1.2) is as follows. We regard (x, y) and (\bar{x}, \bar{y}) as the coordinates of the points P and \bar{P} , respectively, each referred to the same axes Ox, Oy . Then Eqs. (1.1.2) accomplish the rotation of all points of the plane about O clockwise through the angle a .

1.1.3 Galilean transformation

Everyone travelling by train can observe the duality between uniform motions of his local frame of reference (a train) and outside points (people or other objects on a depot). This remarkable exhibition of the duality, when one cannot determine who is actually moving, is known in the classical mechanics as *Galileo's relativity principle*. It is equivalent to the invariance of equations of motion of mechanical systems under the transformation

$$\bar{t} = t, \quad \bar{\mathbf{x}} = \mathbf{x} + t\mathbf{V}, \quad (1.1.5)$$

where \mathbf{V} is the constant velocity. Differentiation of $\bar{\mathbf{x}}$ with respect to $\bar{t} = t$ yields

$$\bar{\mathbf{v}} = \mathbf{v} + \mathbf{V}. \quad (1.1.6)$$

The transformation (1.1.6) of the velocity is a mathematical expression of Galileo's relativity principle. The transformation (1.1.5) is known as the Galilean transformation and lies at the core of the *Galilean group* which is one of the most important groups in non-relativistic physics.

1.2 Introduction of transformation groups

1.2.1 Definitions and examples

We will consider invertible transformations in an n -dimensional Euclidean space \mathbb{R}^n defined, in coordinates, by equations of the form

$$\bar{x}^i = f^i(x), \quad i = 1, \dots, n, \quad (1.2.1)$$

where the vector-function $f = (f^1, \dots, f^n)$ is continuous together with its derivatives involved in further discussions. Since the transformation (1.2.1) is invertible, there exists the inverse transformation

$$x^i = (f^{-1})^i(\bar{x}), \quad i = 1, \dots, n. \quad (1.2.2)$$

Let us denote the transformation (1.2.1) by T and its inverse (1.2.2) by T^{-1} . Thus, T carries any point

$$x = (x^1, \dots, x^n) \in \mathbb{R}^n$$

into a new position

$$\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \mathbb{R}^n,$$

and T^{-1} returns \bar{x} into the original position x . It is assumed that the coordinates x^i and \bar{x}^i of points x and \bar{x} , respectively, are referred to one and the same coordinate system. The identical transformation

$$\bar{x}^i = x^i, \quad i = 1, \dots, n, \quad (1.2.3)$$

will be denoted by I .

Let T_1 and T_2 be two transformations of the form (1.2.1) with functions f_1^i and f_2^i , respectively. Their *product* $T_2 T_1$ (termed also *composition* and denoted by $T_2 \circ T_1$) is defined as the consecutive application of these transformations and is given by

$$\bar{\bar{x}}^i = f_2^i(\bar{x}) = f_2^i(f_1(x)), \quad i = 1, \dots, n. \quad (1.2.4)$$

The geometric interpretation of the product is as follows. Since T_1 carries the point x to the point $\bar{x} = T_1(x)$, which T_2 carries to the new position $\bar{\bar{x}} = T_2(\bar{x})$, the effect of the product $T_2 T_1$ is to carry x directly to its final location $\bar{\bar{x}}$, without a stopover at \bar{x} . Thus, equation (1.2.4) means that

$$\bar{\bar{x}} \stackrel{\text{def}}{=} T_2(\bar{x}) = T_2 T_1(x). \quad (1.2.5)$$

In this notation, the definition of the inverse transformation (1.2.2) means

$$T T^{-1} = T^{-1} T = I. \quad (1.2.6)$$

Definition 1.1. A set G of transformations (1.2.1) in \mathbb{R}^n containing the identity I is called a transformation group if it contains the inverse T^{-1} of every transformation $T \in G$ and the product $T_1 T_2$ of any transformations $T_1, T_2 \in G$. Thus, the attributes of the group G are:

$$I \in G, \quad \text{and} \quad T^{-1} \in G, \quad T_1 T_2 \in G \quad \text{whenever } T, T_1, T_2 \in G. \quad (1.2.7)$$

Example 1.1. The set $G = \{I, T_1, \dots, T_5\}$ of the transformations

$$\begin{aligned} I: \bar{x} = x, \quad T_1: \bar{x} = 1 - x, \quad T_2: \bar{x} = \frac{1}{x}, \\ T_3: \bar{x} = \frac{1}{1-x}, \quad T_4: \bar{x} = \frac{x}{x-1}, \quad T_5: \bar{x} = \frac{x-1}{x} \end{aligned} \quad (1.2.8)$$

on the straight line is a group containing six elements (see, e.g., [6], §9). The group properties (1.2.7) can be verified by computing the inverses and products of the transformations (1.2.8), e.g.

$$\begin{aligned} T_1^{-1} = T_1, \quad T_2^{-1} = T_2, \quad T_3^{-1} = T_5, \quad T_4^{-1} = T_4, \quad T_5^{-1} = T_3, \\ T_1^2 = I, \quad T_2^2 = I, \quad T_3^2 = T_5, \quad T_4^2 = I, \quad T_5^2 = T_3, \\ T_2 T_1 = T_3, \quad T_1 T_2 = T_5, \quad T_3 T_1 = T_2, \quad T_1 T_3 = T_4. \end{aligned} \quad (1.2.9)$$

Example 1.2. Consider the set G of all translations (displacements) T_a :

$$\bar{x} = x + a \quad (1.2.10)$$

on the straight line. Since $\bar{x} = x$ when $a = 0$, the set G contains the identity $I = T_0$. Furthermore, the combined effect of two translations, T_a and T_b , acting in succession, is to displace x through the distance $a + b$. Hence,

$$T_b T_a = T_{a+b}. \quad (1.2.11)$$

Equation (1.2.11) shows that

$$T_a^{-1} = T_{-a}.$$

Thus, the transformations (1.2.10) obey the group properties (1.2.7), and hence define a *one-parameter group* G , i.e. a group containing one arbitrary parameter a . This group is known as the *translation group* and provides one of the simplest illustrations to the following definition.

Definition 1.2. A set G of transformations T_a in \mathbb{R}^n depending continuously on a parameter a , where a ranges over all real numbers from a given interval $U \subset \mathbb{R}$, is called a *one-parameter group* if there is a unique value $a = a_0$ in U providing the identical transformation, $T_{a_0} = I$, and the following conditions hold for all $a, b \in U$:

$$T_a^{-1} = T_{a^{-1}} \in G, \quad T_b T_a = T_c \in G, \quad (1.2.12)$$