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Optimal Transportation

Theory and Applications

Edited by
Yann Ollivier, Hervé Pajot and Cédric Villani

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Preface

This book contains the proceedings of the summer school "Optimal transportation: Theory and Applications" held at the Fourier Institute (University of Grenoble I, France). The first 2 weeks were devoted to courses that described the main properties of optimal transportation and discussed its applications to analysis, differential geometry, dynamical systems, partial differential equations and probability theory. Courses were addressed both to students and researchers. A workshop took place during the last week. The aim of this conference was to present very recent developments of optimal transportation and also its applications in biology, mathematical physics, game theory and financial mathematics.

The first part of the book contains (expanded) versions of the courses. There are two sets of notes by F. Santambrogio. The first one gives a short introduction to optimal transport theory. In particular, the Kantorovich duality, the structure of Wasserstein spaces and the Monge–Ampère equations related to optimal transport are presented to the readers. These notes could be seen as an introduction for the other papers of the book. The second one describes applications to economics, game theory and urban planning.

The notes of I. Gentil, P. Topping and S.-I. Ohta describe (with different flavours) the connections between optimal transport and the notion of Ricci curvature, which is a very important tool in classical Riemannian geometry. A notion of curvature-dimension condition was defined by D. Bakry and M. Émery to study geometric properties of diffusions and to get functional inequalities. I. Gentil's notes study the Bakry–Émery condition in the case of the Ornstein–Uhlenbeck semigroup. A quite different approach using optimal transport theory to obtain logarithmic Sobolev-type inequalities is also discussed. A definition of metric measure spaces with lower Ricci curvature bound (which coincides with the classical definition in the case of Riemannian manifolds) was proposed very recently by K.-T. Sturm and J. Lott–C.

Villani independently. S.-I. Ohta's long paper discusses in detail the geometry of such spaces. For instance, versions of the Brunn–Minkowski inequality, of the Lichnerowicz inequality, of Bishop–Gromov volume comparison, of the Bonnet–Myers diameter bound and also the stability under Hausdorff–Gromov convergence are proved in this general setting. The theory of Ricci flow as developed by R. Hamilton and others since 1982 is an essential element in the proofs by G. Perelman of the Poincaré conjecture and Thurston's geometrisation conjecture. The objective of P. Topping's lectures is to explain this theory from the point of view of optimal transport.

The fundamental work of Y. Brenier related to the Euler equation played an important role in the renewal of optimal transport in the 1980s. The notes of L. Ambrosio and A. Figalli describe some recent results on Brenier's variational models for the incompressible Euler equations.

The paper by S. Daneri and G. Savaré gives an overview of the theory of gradient flows in Euclidean spaces and then in metric spaces. Applications to evolution equations in the Wasserstein spaces of probability measures are also discussed.

Apart from these mini-courses, this book also contains five research/survey papers. O. Besson, M. Picq and J. Pousin present an algorithm for a computing mass transport problem inspired from optimal transport and whose origin lies in hearts' images tracking. M. Beigblöck, C. Léonard and W. Schachermayer discuss the duality theory for the Monge–Kantorovich transport problem. In particular, they give a version of Fenchel's perturbation method. The paper of F. Bolley reviews recent quantitative results on the approximation of mean field diffusion equations by large systems of interacting particles, obtained by optimal coupling methods. P. Cattiaux and A. Guillin describe some recent results on Poincaré-type inequalities, transportation-information inequalities or logarithmic Sobolev inequality obtained via Lyapounov conditions. Q. Mérigot proves the stability of the Federer curvature measures with respect to the Wasserstein distance. This was motivated by problems of reconstruction of curves and surfaces from point cloud approximation that come from image analysis for instance. These five contributions illustrate the variety of possible applications of optimal transport to pure and applied mathematics, and also to computer science.

Contents

PART ONE

Short Courses

1

Introduction to optimal transport theory

FILIPPO SANTAMBROGIO

Abstract

These notes constitute a sort of crash course in optimal transport theory. The different features of the problem of Monge–Kantorovich are treated, starting from convex duality issues. The main properties of space of probability measures endowed with the distances W_p induced by optimal transport are detailed. The key tools connecting optimal transport and partial differential equations are provided.

Contents

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1.1 Introduction

These very short lecture notes are not intended to be an exhaustive presentation of the topic, but only a short list of results, concepts and ideas which are useful when dealing for the first time with the theory of optimal transport. Several of these ideas have been used, and explained in greater detail, during the other classes of the Summer School "Optimal Transportation: Theory and Applications" which were the occasion for the redaction of these notes. The style that was chosen when preparing them, in view of their use during the Summer School, was highly informal, and this revised version will respect the same style.

The main references for the whole topic are the two books on the subject by C. Villani [15, 16]. For what concerns curves in the space of probability measures, the best specifically focused reference is [2]. Moreover, I am also very indebted to the approach that L. Ambrosio used in a course at SNS Pisa in 2001–02 and I want to cite this as another possible reference [1].

The motivation for the whole subject is the following problem proposed by Monge in 1781 [14]: given two densities of mass $f,\ g \geq 0$ on $\mathbb{R}^d$, with $\int f = \int g = 1$, find a map $T : \mathbb{R}^d \to \mathbb{R}^d$ pushing the first one onto the other, i.e. such that

$$\int_A g(x)dx = \int_{T^{-1}(A)} f(y)dy \quad \text{for any Borel subset } A \subset \mathbb{R}^d \tag{1.1}$$

and minimizing the quantity

$$\int_{\mathbb{R}^d} |T(x) - x| f(x)dx$$

among all the maps satisfying this condition. This means that we have a collection of particles, distributed with density f on $\mathbb{R}^d$, that have to be moved, so that they arrange according to a new distribution, whose density is prescribed and is g. The movement has to be chosen so as to minimize the average displacement. The map T describes the movement (that we must choose in an optimal way), and $T(x)$ represents the destination of the particle originally located at x. The constraint on T precisely accounts for the fact that we need to reconstruct the density g. In the following, we will always define, similarly to (1.1), the image measure of a measure μ on X (measures will indeed replace the densities f and g in the most general formulation of the problem) through a measurable map $T : X \to Y$: it is the measure denoted by $T_{\#}\mu$ on Y and characterized by

$$T_{\#}\mu(A) = \mu(T^{-1}(A)) \quad \text{for every measurable set } A,$$
$$\text{or} \int_Y \phi \, d(T_{\#}\mu) = \int_X \phi \circ T \, d\mu \quad \text{for every measurable function } \phi.$$

The problem of Monge has stayed with no solution (Does a minimizer exist? How to characterize it?...) until the progress made in the 1940s. Indeed, only with the work by Kantorovich in 1942 has it been inserted into a suitable framework which gave the possibility to approach it and, later, to find that solutions actually exist and to study them. The problem has been widely generalized, with very general cost functions $c(x, y)$ instead of the Euclidean distance $|x - y|$ and more general measures and spaces. For simplicity, here we will not try to present a very wide theory on generic metric spaces, manifolds and so on, but we will deal only with the Euclidean case.

1.2 Primal and dual problems

In what follows we will suppose Ω to be a (very often compact) domain of $\mathbb{R}^d$ and the cost function $c : \Omega \times \Omega \to [0, +\infty[$ will be supposed continuous and symmetric (i.e. $c(x, y) = c(y, x)$).

1.2.1 Kantorovich and Monge problems

The generalization that appears as natural from the work of Kantorovich [12] of the problem raised by Monge is the following:

Problem 1. Given two probability measures μ and ν on Ω and a cost function $c : \Omega \times \Omega \to [0, +\infty]$ we consider the problem

$$(K) \quad \min \left\{ \int_{\Omega\times\Omega} c \, d\gamma \,|\, \gamma \in \Pi(\mu, \nu) \right\}, \tag{1.2}$$

where $\Pi(\mu, \nu)$ is the set of the so-called *transport plans*, i.e. $\Pi(\mu, \nu) = \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (p^+)_{\#}\gamma = \mu,\ (p^-)_{\#}\gamma = \nu\}$, where p^+ and p^- are the two projections of $\Omega \times \Omega$ onto Ω. These probability measures over $\Omega \times \Omega$ are an alternative way to describe the displacement of the particles of μ: instead of saying, for each x, which is the destination $T(x)$ of the particle originally located at x, we say for each pair (x, y) how many particles go from x to y. It is clear that this description allows for more general movements, since from a single point x particles can a priori move to different destinations y. If multiple destinations really occur, then this movement cannot be described through a map T. Notice that the constraints on $(p^{\pm})_{\#}\gamma$ exactly mean that we restrict our attention to the movements that really take particles distributed according to the distribution μ and move them onto the distribution ν.

The minimizers for this problem are called *optimal transport plans* between μ and ν. Should γ be of the form $(id \times T)_{\#}\mu$ for a measurable map $T : \Omega \to \Omega$ (i.e. when no splitting of the mass occurs), the map T would be called an *optimal transport map* from μ to ν.

Remark 1. It can be easily checked that if $(id \times T)_{\#}\mu$ belongs to $\Pi(\mu, \nu)$ then T pushes μ onto ν (i.e. $\nu(A) = \mu(T^{-1}(A))$ for any Borel set A) and the functional takes the form $\int c(x, T(x))\mu(dx)$, thus generalizing Monge's problem.

This generalized problem by Kantorovich is much easier to handle than the original one proposed by Monge, for instance, in the Monge case we would need existence of at least a map T satisfying the constraints. This is not verified when $\mu = \delta_0$, if ν is not a single Dirac mass. On the contrary, there always exists a transport plan in $\Pi(\mu, \nu)$ (for instance, $\mu \otimes \nu \in \Pi(\mu, \nu)$). Moreover, one can state that (K) is the relaxation of the original problem by Monge: if one considers the problem in the same setting, where the competitors are transport plans, but sets the functional at $+\infty$ on all the plans that are not of the form $(id \times T)_{\#}\mu$, then one has a functional on $\Pi(\mu, \nu)$ whose relaxation is the functional in (K) (see [3]).

Anyway, it is important to notice that an easy use of the direct method of calculus of variations (i.e. taking a minimizing sequence, saying that it is compact in some topology – here it is the weak convergence of probability measures – finding a limit, and proving semicontinuity (or continuity) of the functional we minimize, so that the limit is a minimizer) proves that a minimum does exist.

As a consequence, if one is interested in the problem of Monge, the question may become "Does this minimum come from a transport map T?" Actually, if the answer to this question is yes, then it is evident that the problem of Monge has a solution, which also solves a wider problem, that of minimizing among transport plans. In some cases, proving that the optimal transport plan comes from a transport map (or proving that there exists at least one optimal plan coming from a map) is equivalent to proving that the problem of Monge has a solution, since very often the infimum among transport plans and among transport maps is the same. Yet, in the presence of atoms, this is not always the case, but we will not insist any more on this degenerate case.