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CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 22

Representations and characters of finite groups

To Marjorie

Preface

Representation theory and character theory provide major tools for the study of finite groups. Complex representations and their characters were first studied nearly 100 years ago by Frobenius, and his theorem on transitive permutation groups was the first major achievement of the theory; it remains to this day, along with Burnside's $p^a q^b$-theorem, one of the highlights of any first course on representation theory. Indeed, while Burnside's theorem now admits a purely group theoretic proof, Frobenius' theorem remains untouched by non-character-theoretic methods.

Both of these results may be regarded as nonsimplicity criteria. The study and application of character theory, since Brauer proposed a systematic programme to classify the finite simple groups at the Amsterdam International Congress in 1954, cannot be divorced from the classification itself. Although purely group theoretic methods have dominated the major part of that work which took place between 1970 and 1980, the classification could never have been carried out (and, indeed, would have been stillborn) without the early progress made using character theory. The reason for this is quite simple. The goal is always to obtain global information from local information (that is, information about the whole group from information about various subgroups). However, the process is inductive; when a group is in some sense big enough, there is enough interaction between the various subgroups to progress from that interaction by group theoretic means, but in small configurations the information available is so tight (and the situation which occurs in the proof of Frobenius' theorem is a perfect example) that what Brauer referred to as the 'arithmetic' properties of groups, namely their characters, have to be studied. Thus characters have played an indispensable role in the study of certain permutation groups and in the characterisation of groups with Sylow 2-subgroups of small rank (possibly zero), and this will remain necessary in the context of the current revision project.

There have been two major strands to this work; on the one hand, there is Brauer's theory of blocks of characters, arising out of his work on modular representation theory, and, on the other, the theory of exceptional characters developed by Suzuki, Feit and others, as a direct attempt to generalise the original ideas of Frobenius. The aim of this book is to give a comprehensive and self-contained account of the latter, with all the refinements and improvements which are now possible.

The origin of this book lies in graduate lectures that I gave in Oxford over the years 1971–3 and again in 1976–8, together with other courses given since. In the original courses, I covered both the ordinary (complex) theory and modular theory but, with an exception on which I shall comment below, I have concentrated here on those areas of ordinary representation theory and character theory which should occur in any graduate course or which will serve as an introduction for those who wish to study their application to the classification of simple groups, and I have included a substantial amount of material which I did not put into any of the lectures, and some of which is far more recent. In a book of this length, one cannot hope to cover all topics and I have made no such attempt; for example, there is no mention of the character theory of soluble groups or the symmetric groups, nor any of Schur indices beyond the basic Frobenius–Schur index. On the other hand, I have written this book very much for the group theorist so that, for example, the first chapter contains a considerable amount of general material that is relevant to the application of representation theoretic methods to 'internal' group theory, and this book contains sufficient for anyone wishing to go on to study the character theory of the odd order paper of Feit and Thompson, either in the original form or as revised by Sibley.

This book should be accessible both to graduates and to higher level undergraduates, and the range of topics and exercises in the first two chapters which cover basic representation theory and character theory will have such readers in mind, though a judicious choice of material for undergraduate use might be desirable. Only a very basic knowledge of group theory and more general algebra is assumed, though a more sophisticated background would be valuable to see some of the examples and applications in their full perspective; unavoidably, some of the material in these chapters is there for later application. I have, however, tried to prepare the reader for study in areas other than those covered later; there is sufficient background from the representation theory side, for example, to study the Deligne–Lusztig theory of the ordinary characters of groups of Lie type. Also, in the final section of Chapter 2, I have included a brief discussion of the application of the character theoretic methods which are central in this book to the Inverse Galois problem.

After the first two chapters, our treatment becomes more specialised. In Chapter 3, we examine Suzuki's theory of exceptional characters and then, in Chapter 4, Feit's theory of isometries and coherence. Sibley's refinements of Feit's work are discussed here and they are applied to simplify two major applications of character theory; in Section 4.5, we shall establish the nonsimplicity of CN-groups of odd order following Suzuki's treatment of CA-groups of odd order, and, in Section 4.6, we shall give a unified treatment of the reduction theorems for Zassenhaus groups which were originally proved by Feit and by Itô. In Chapter 5, which is self-contained, Brauer's characterisation of characters is discussed; this topic may today be included in any basic course and can be read immediately after Section 2.3, but here is the appropriate place in the context of this book since we wish to emphasise its role in the construction of isometries other than that arising directly from character induction as in the Suzuki theory. This work takes place in the final chapter.

It is in this final chapter that I make the one exception to self-containment. Block theory is needed for Reynolds' work and for the more recent work of Robinson on isometries, for which I have given a unified treatment. I had intended just to state the most basic results required but, as I was writing, I found that all but Brauer's second and third main theorems (which are only stated) can be proved by methods already discussed at length, and I have therefore indulged in a somewhat unusual approach to block theory. That this should prove possible did not come as a surprise. I have long felt that Brauer's method of columns was somehow the dual of Suzuki's method of exceptional characters (and this is implicit in work of Walter and Wong), and I first gave a proof of the nonsimplicity of groups with homocyclic Sylow 2-subgroups of rank 2, replacing the use of Brauer's method of columns by an isometry constructed using Brauer's characterisation of characters, in my lectures; what did surprise me was that I could also construct the principal 2-block for groups with dihedral Sylow 2-subgroups by the same approach. This has allowed me to give in a natural way Suzuki's proof of the Brauer–Suzuki theorem on groups with an ordinary quaternion Sylow 2-subgroup.

The link is Brauer's characterisation of characters, and I have chosen to give the Brauer–Tate proof rather than the shorter proof of Goldschmidt and Isaacs since I believe that it may give greater insight into the relationship between character ring methods and Brauer's methods with modular characters. My one regret is that I cannot justify this comment explicitly; although modular representation theory is very much in vogue today, I would hope that Brauer's second and third main theorems will one day submit to non-modular proofs.

Finally I turn to acknowledgements. I owe my own original interest in character theory to some second year undergraduate lectures 'advertising' algebra which were given in Oxford by Martin Powell some 25 years ago and to the course given by Graham Higman when I was first his research

student; Section 3.4, in particular, reflects this upbringing, although Theorem 3.17 turns out to have a new application in my treatment of Zassenhaus groups! Looking back to those formative years, I should also thank Michio Suzuki who sent me a number of his papers, one of which contains the marginal note, 'More work is needed'.

For the metamorphosis of my lectures into this book, I would like to thank several generations of students in Oxford (and one at CalTech) who have unearthed many unclear points (and a few howlers). Jonathan Alperin, Peter Landrock and, in particular, Geoffrey Robinson have read sections of the manuscript at various stages, and I am grateful for all their comments. It would also be appropriate to thank Jack Cowan who introduced me to the Macintosh; without it, I would not have been able so easily to send draft sections around for comment and make almost constant changes to the manuscript. I should also like to thank the Universities of Aarhus, Chicago and Essen for their hospitality and support at various times while this book was being written.

Finally I should like to thank all those at the Cambridge University Press for their assistance – and, in particular, David Tranah for his patience.

MICHAEL COLLINS

Oxford
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Contents

1

General representation theory

1 Basic concepts

Let G be an arbitrary finite group[†] and let K be an arbitrary field. Then a *(linear) representation ρ of G over K* is a homomorphism

$$\rho: G \to \mathrm{GL}(V)$$

where V is a finite dimensional vector space[†] over K and $\mathrm{GL}(V)$ is the group of nonsingular linear transformations of V into itself. If we are already given the vector space V, then we may refer to ρ as a representation of *G on V*. Although we shall normally write mappings on the right, we shall write $\rho(g)$ rather than $g\rho$ since we shall never consider compositions of representations; also, for $v \in V$, we shall write vg for $v \cdot \rho(g)$ if there is no risk of ambiguity.

If the dimension of V is n, we may choose a basis for V and identify V with the space K^n of n-tuples over K; then we may regard ρ as a map from G into $\mathrm{GL}(n, K)$, the group of nonsingular $n \times n$ matrices over K. The precise map so obtained depends on the choice of basis; thus a homomorphism

$$\rho: G \to \mathrm{GL}(n, K)$$

should be called a *matrix* representation. However, in a way which will be made precise shortly, matrix representations obtained by taking different bases are *similar*, and we shall move freely between representations on vector spaces and the corresponding matrix representations.

We shall study representations with two particular purposes in mind. The first is that a representation gives us something concrete, namely a group of linear transformations or matrices, to which the methods of linear algebra may be applied. The second is that by studying the values of the traces of the matrices $\rho(g)$, it may be possible to use the arithmetic properties of the field K to deduce information about an abstract group G. This is known as *character theory*, and much of this book will be devoted to this aspect in the case that K is the field of complex numbers $\mathbb{C}$.

[†] Throughout this book, groups will always be finite, with the obvious exception of groups of linear transformations, and vector spaces will be finite dimensional. However, most of the definitions of this section, although little of the subsequent theory, can be extended without these restrictions.

Such trace values are known as (ordinary) *characters*. However, in this chapter we shall develop the basic representation theory in the first spirit and in a form more general than that needed purely for character theory.

Examples (Groups and fields are arbitrary unless otherwise stated.)

1. Let V be a one-dimensional vector space over K. The map

$$g \to 1_V$$

for all $g \in G$ is the *trivial* representation of G over K.

2. Let G be a group which acts as a group of permutations on a finite set Ω, where $\Omega = \{e_1, \ldots, e_n\}$. Let V be a vector space of dimension n over K with a basis $\{v_1, \ldots, v_n\}$. For $g \in G$, let π_g be the linear transformation on V defined by the action on basis vectors

$$\pi_g : v_i \to v_j \quad \text{if and only if} \quad g : e_i \to e_j.$$

Then the map $\pi : G \to \mathrm{GL}(V)$ defined by $\pi(g) = \pi_g$ for all $g \in G$ is a *permutation* representation of G on V. Notice that the corresponding matrix representation (with respect to the basis $\{v_1, \ldots, v_n\}$) is given by permutation matrices.

3. Take $\Omega = G$ in Example 2 and define a permutation action by the mappings

$$g : x \to xg$$

for all $x, g \in G$. The associated representation is called the *right regular representation* of G.

4. Let N be a normal subgroup of G, and suppose that ρ is a representation of G/N. The mapping

$$\hat{\rho} : g \to \rho(gN)$$

for all $g \in G$ defines a representation on G. This representation is called the *inflation* of ρ.

Conversely, if σ is a representation of G such that N lies in the kernel of σ, then the mapping

$$\tilde{\sigma} : gN \to \sigma(g)$$

defines a representation of G/N.

5. If K is regarded as a one-dimensional vector space over itself, then multiplication acts as a linear transformation. Thus any homomorphism from a group G into the multiplicative group of K may be viewed as a representation. In particular, if ρ is a representation of G over K, then

the mapping

$$g \to \det(\rho(g))$$

for all $g \in G$ is a one-dimensional representation.

6. Let G be a cyclic group of order n and let g be a generator of G. Let ω be an nth root of unity in K (not necessarily primitive). Then the mapping

$$\rho_\omega : g^i \to \omega^i$$

defines a representation of G. Conversely, every one-dimensional representation of G is similar to a representation of this form.

7. The alternating groups A_4 and A_5 and the symmetric group S_4 are isomorphic, respectively, to the rotation groups of the regular tetrahedron, icosahedron and cube. By taking an orthonormal basis for $\mathbb{R}^3$, these isomorphisms lead to natural representations of the three groups by real *orthogonal* 3×3 matrices.

We shall now introduce some basic terminology. Let G and K be, as before, arbitrary and let ρ be a representation of G on a vector space V over K. The dimension $\dim_K(V)$ is called the *degree* of the representation ρ and will be denoted by $\deg \rho$. If the kernel, $\ker \rho$, of ρ is trivial, then ρ is *faithful.* If U is a subspace of V which is invariant under $\rho(g)$ for all $g \in G$, then U *admits* G, or is *G-invariant.* If $V \neq 0$ and the only G-invariant subspaces of V are 0 and V itself, then ρ is *irreducible*; otherwise ρ is *reducible.* If V can be written as the direct sum of two nonzero G-invariant subspaces, then ρ is *decomposable*; otherwise ρ is *indecomposable.*

It follows, trivially, that an irreducible representation is indecomposable. The converse is true provided that the characteristic of K does not divide† the order of G as we shall see in Section 3, but this need not be so in general. For example, a two-dimensional representation of the additive group of $\mathbb{Z}_p$ over $\mathbb{Z}_p$ is given by

$$t \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and this is indecomposable but not irreducible: the subspace spanned by the second basis vector is invariant, but not complemented.

Suppose that ρ_1 and ρ_2 are two representations of G over K on vector spaces V_1 and V_2 respectively. Then ρ_1 and ρ_2 are said to be *equivalent* if there exists an isomorphism $\sigma: V_1 \to V_2$ such that

$$\rho_2(g) = \sigma^{-1} \rho_1(g) \sigma \quad \text{for all } g \in G;$$

we shall write $\rho_1 \sim \rho_2$, or $\rho_1 \sim_K \rho_2$ if we wish to emphasise the field K.

†This will always be understood to include the case that char $K = 0$.

Visibly, this defines an equivalence relation on the representations of a group (over a fixed field), and by a set of *distinct* representations of a group, we shall always mean a collection of inequivalent representations.

If $V_1 = V_2$, then this definition can be applied to two different representations of G on V_1; also, if ρ is a representation of G on V and ρ_1 and ρ_2 are the associated *matrix* representations with respect to different bases of V, then immediately ρ_1 and ρ_2 are equivalent. In this case, there exists a nonsingular matrix X such that

$$\rho_2(g) = X^{-1}\rho_1(g)X$$

for all $g \in G$, and we say that ρ_1 and ρ_2 are *similar*.

Exercises

1. Show that the derived group G' of a group G lies in the kernel of any representation of G of degree 1. Deduce that, if $\rho: G \to \mathrm{GL}(n, K)$ is a matrix representation of G, then $\rho(g) \in \mathrm{SL}(n, K)$ whenever $g \in G'$.
2. Let ρ_1 and ρ_2 be equivalent representations of a group G. Show that, whenever $g \in G$, the linear transformations $\rho_1(g)$ and $\rho_2(g)$ have the same minimal and characteristic polynomials. If g is an element of order n, show that the minimal polynomial of $\rho_1(g)$ divides $x^n - 1$.
3. Let ρ be a representation of a group G over an algebraically closed field K whose characteristic does not divide $|G|$. If g is a fixed element of G, show that there exists a basis with respect to which $\rho(g)$ has a diagonal matrix.
4. Let G be a finite abelian group and let K be an algebraically closed field of characteristic not dividing $|G|$. If G has a representation on a vector space V over K, show that there exists a basis for V with respect to which every element of G is represented by a diagonal matrix. Deduce that every irreducible representation of G over K has degree 1.
5. Let G and K be as in Exercise 4 and regard the irreducible representations as maps from G to K. Suppose that G has a decomposition as the direct product of cyclic subgroups generated by elements $g_1, \ldots, g_r$. Show that an irreducible representation ρ of G is determined by its values on the elements $g_1, \ldots, g_r$ alone. Deduce that the number of distinct irreducible representations of G over K is $|G|$.

   Show that the set of distinct irreducible representations forms an abelian group G^* under composition defined by

   $$(\rho_1 \cdot \rho_2)(g) = \rho_1(g) \cdot \rho_2(g) \quad \text{for all } g \in G,$$

   and that G^* is isomorphic (as an abstract group) to G.

6. Show that an irreducible representation of a cyclic group G of prime order p over a field of characteristic p is trivial. By considering the possible Jordan canonical forms for a linear transformation of order p, determine a complete set of inequivalent indecomposable representations of G over a field of characteristic p.
7. Determine the irreducible representations of an arbitrary p-group over a finite field K of characteristic p.
   [Hint. Show that the subgroup of upper triangular matrices in $\mathrm{GL}(n, K)$ is a Sylow subgroup, where n is the degree, and apply Sylow's theorem.]
8. Let G be the dihedral group D_{2n} of order $2n$, the group of symmetries of a regular n-gon. Then G has a presentation
$$G = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle.$$
   Suppose that n is odd. Show that $G' = \langle x \rangle$, and hence determine the (two) one-dimensional representations of G over $\mathbb{C}$.
   Use Exercises 1 and 4 to show that, if $\rho: G \to \mathrm{GL}(2, \mathbb{C})$ is a two-dimensional irreducible complex representation of G, then $\rho \sim \rho_1$ where
$$\rho_1(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$
   and ω is a nonidentity nth root of unity. Determine which matrices of order 2 can invert $\rho_1(x)$, and hence show that $\rho_1 \sim \rho_2$ where
$$\rho_1(x) = \rho_2(x) \quad \text{and} \quad \rho_2(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
   Deduce that the number of inequivalent irreducible complex representations of G of degree 2 is $\frac{1}{2}(n-1)$.
   [Notice that $\frac{1}{2}(n-1)\cdot 2^2 + 2\cdot 1^2 = 2n$: see Exercise 17 of Section 2 and also Corollary 20 (iii).]
9. Carry out the corresponding analysis to Exercise 8 when n is even. [Note that, in this case, $G' = \langle x^2 \rangle$.]
10. Let G be the generalised quaternion group of order 2^{n+1} ($n \geqslant 2$) which has a presentation
$$g = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle.$$
    Show that there is a complex representation $\rho: G \to \mathrm{GL}(2, \mathbb{C})$ for which
$$\rho(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
    where ω is a primitive 2^nth root of unity and that ρ is faithful and

irreducible. Show also that every element of G is represented by a matrix in $SL(2,\mathbb{C})$, and deduce directly that G contains a unique involution (element of order 2).

11. In Example 7, use geometrical considerations to show that the representations defined there are irreducible.
12. Let G be a group, and define an action of G as a group of permutations on itself by $g: x \to g^{-1}x$. Show that this gives rise to a representation over any field, called the *left regular representation.*
13. Show that, if ρ and σ are similar matrix representations of a group G over a field K, then $\operatorname{tr}(\rho(g)) = \operatorname{tr}(\sigma(g))$ for all $g \in G$.

    Show also that $\operatorname{tr}(\rho(g)) = \operatorname{tr}(\rho(h))$ whenever g and h are conjugate elements in G. (This says that we have defined a *class function* on G.) [tr denotes the *trace* of matrix: if $A = (a_{ij})$, then $\operatorname{tr} A = \sum_i a_{ii}$.]
14. For each of the groups A_4, A_5 and S_4, determine the value of $\operatorname{tr}(\rho(g))$ for a representative of each conjugacy class, where ρ is the three-dimensional real representation defined for each of the three groups in Example 7.

2 Group rings, algebras and modules

Let G be a finite group and let ρ be a representation of G on a vector space V over a field K. Then the K-linear combinations of the linear transformations $\rho(g)$ for $g \in G$ form a subring of the full ring $\mathscr{L}(V)$ of linear transformations of V. The vector space V can be given the structure of a right module over this subring. We shall formalise this, but make our first definition more general.

Let G be a group and let R be a commutative ring with identity. Then the *group ring RG* consists of the set of all formal sums

$$\sum_{g \in G} a_g g \quad (a_g \in R)$$

together with the binary operations

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g \quad (a_g, b_g \in R)$$

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{h\in G} b_h h\right) = \sum_{g\in G}\left(\sum_{h\in G} a_{gh^{-1}} b_h\right) g$$

$$= \sum_{g,h\in G} (a_g b_h)(gh)$$

where gh is the group product in G. It is a straightforward calculation to verfiy that RG is an associative ring with identity. If R is a field K, then KG has the structure of a vector space over K as well as that of ring. So