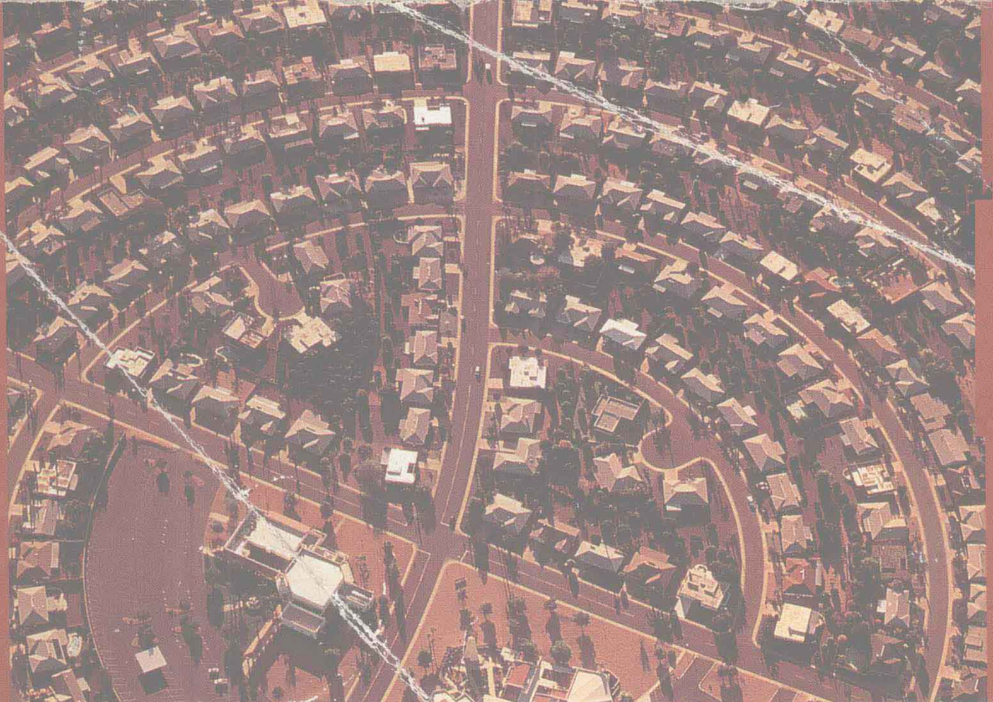

DAVID A. SMITH / LAWRENCE C. MOORE

CALCULUS

Modeling and Application

Calculus

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David A. Smith Lawrence C. Moore
Duke University

D. C. HEATH AND COMPANY
Lexington, Massachusetts Toronto

Address editorial correspondence to:
D. C. Heath and Company
125 Spring Street
Lexington, MA 02173

Acquisitions: Charles Hartford
Development: Kathleen Sessa-Federico
Editorial Production: Melissa Ray and Craig Mertens
Design: Cornelia Boynton
Art Editing: Gary Crespo
Production Coordination: Charles Dutton
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Preface

Calculus is the study of change. The concepts of calculus enable us to model processes that change and to describe properties of these processes that remain constant in the midst of change. Now change has come to the **learning** of calculus — change driven by the need to respond to the revolution in technology and by our increased awareness of how students learn. This text is an outgrowth and agent of that change. We wrote this book initially to support the reformed calculus course developed at Duke University with the support of the National Science Foundation. Since 1990, various preliminary versions have been used at many other colleges across the country. The experiences of dozens of teachers and many thousands of students have helped us refine our work to the form of this first edition.

In our development of the course and the text we were guided by the following goals:

- Students should be able to use mathematics to structure their understanding of and investigate questions in the world around them.
- Students should be able to use calculus to formulate problems, to solve problems, and to communicate their solutions of problems to others.
- Students should be able to use technology as an integral part of this process of formulation, solution, and communication.
- Students should work and learn cooperatively.

The course we developed to serve these goals emphasizes

- real-world problems,
- hands-on activities,
- discovery learning,
- writing and revision of writing,
- teamwork,
- intelligent use of available tools, and
- high expectations of students.

Our *Instructor's Guide* explains how the various components of the course fit together and suggests a variety of ways it can be taught — most of them based on actual experience at one or more campuses. In this Preface we concentrate on how the book itself serves our goals and emphases.

Real-world contexts We provide a real-world setting for each concept and calculational rule. For example, both differentiation and the exponential function appear early in Chapter 2 in connection with natural growth of populations. The Chain Rule is introduced as part of the modeling process for reflection and refraction of light. Both improper integrals and polynomial approximation of functions result from an investigation of models for the distribution of data. Throughout, we emphasize differential equations and initial value problems — the main connection between calculus and applications in the sciences and engineering.

Discovery learning There are three features in the book that enhance discovery learning: Exploration Activities, Checkpoints, and Examples. We encourage students to construct their own knowledge by attempting Exploration Activities embedded in the text. These activities invite students to explore new concepts and problems, and to see what the issues are before they are discussed in detail in the text. Each of these activities is followed by a discussion that leads into the new ideas. In addition, we encourage students to check their understanding by attempting Checkpoint calculations, which appear after the appropriate new ideas have been introduced; their answers are given at the ends of the sections. The combination of Exploration Activities, Checkpoints, and Examples provides a wealth of engaging illustrations of the central concepts and techniques.

Problem solving Throughout the text we emphasize general principles and practice in problem solving. At the end of each section and of each chapter we have provided a range of exercises, from practice calculations to additional development of concepts. Answers are provided at the back of the book for approximately one-fourth of the calculational exercises. (If an answer is provided, the exercise number is underlined in the text.) The exercises are intended primarily for individual students, although the more challenging ones also work well for group activities in or out of class. In addition, each chapter has a number of projects, which are more open-ended explorations designed for investigation by small groups of three or four students. They may be used in a variety of ways — for in-class activities that are written up in homework style or as a basis for more formal reports.

Technology We assume each student has access to a graphing calculator, but no particular make or model is assumed. We encourage students to supplement our figures with their own graphs and plots of data, and we help them find ways to use their calculators for checking other work, such as symbol manipulation.

We assume that the solution of equations and definite integration are either buttons on the calculator or available programs. We study Newton's Method, Simpson's Rule, and other numerical techniques to understand how calculators and computers work — in particular, how they use ideas from calculus. In our study of the Fundamental Theorem, we emphasize construction of antiderivatives by definite integration, a process easily carried out by a calculator.

Laboratories We have written this text so that no particular laboratory activity is required — the text stands on its own. However, we expect that many students using this text will participate in more extensive investigations using programmable calculators or computer software. We have developed companion laboratory manuals that motivate, clarify, and deepen student understanding. These manuals, listed at the end of this Preface, are available for a wide range of technologies. Lab activities are an excellent source of writing projects, ranging from fill-in-a-paragraph answers to full-scale reports. At the ends of some chapters in this book there are Optional Lab Readings designed to provide background for particular lab projects. (They can also serve as supplementary readings.)

Chapter Content

In Chapter 1 we study the concept of function in a context of mathematics as a tool for modeling. Because of our emphasis on technology, we also discuss numerical calculation and significant digits in a section that may be unique for a book at this level.

Chapter 2 begins the study of rates of change in the context of exponential growth. Here we introduce the derivative and the natural exponential function. In addition, we take our first look at slope fields and initial value problems. We conclude the chapter with semilog and log-log plots and their uses for discovering growth patterns in data.

In Chapter 3 we take a more detailed look at initial value problems in the context of Newton's Law of Cooling — to solve a murder mystery. We also study falling bodies — first without air resistance, but one of the end-of-chapter projects explores linear resistance and links the falling-body problem to a number of other exponential decay situations, including Newton's Law of Cooling.

In Chapter 4 we obtain the remainder of the basic calculational tools for differential calculus and examine the interplay between function graphs and values of the derivatives. Throughout we raise issues of antidifferentiation along with differentiation. Our emphasis on simple differential equations allows us to address topics needed for physics and engineering courses well in advance of the more difficult topic of the definite integral.

Chapter 5 introduces Euler's Method for numerical solution of initial value problems to examine the SIR-model of epidemics. We also look at both continuous and discrete models for the evolution of prices in a simple economy. Here we see instability in the discrete model, a harbinger of the discussion of chaos in Chapter 7.

In Chapter 6 we study trigonometric functions and their derivatives in the context of periodic motion — modeling springs and pendulums.

Chapter 7 investigates symbolic solutions of separable differential equations, continuing our thread of modeling population growth. Here we obtain symbolic solutions to the logistic equation and investigate a superexponential model for world population growth. The discrete form of the logistic differential equation — equivalent to an Euler's Method approximation to the continuous model — leads to a unit on chaos.

In Chapter 8 we introduce the definite integral and obtain the Fundamental Theorem of Calculus from our understanding of Euler's Method for approximation of the solution of an initial value problem. We stress the role of the Fundamental Theorem in solving problems of antidifferentiation, because calculators can construct functions by definite integration to a variable upper limit. The traditional role for finding definite integrals remains important for problems that depend on parameters, as is often the case with mathematical models.

Chapter 9, on integral calculus, parallels Chapter 4, on differential calculus. Here we investigate some of the uses of the definite integral with emphasis on how one decides that the calculation of a particular integral is what is needed in a given situation. We look at numerical methods for evaluating integrals from the point of view, "What might your calculator or computer be doing when you press that key?" Because not everyone has constant access to a symbolic computer system, we include an extensive section on the use of the Integral Table. Finally, to motivate integrals of trigonometric functions, we include a brief introduction to Fourier approximations for analyzing complex periodic phenomena.

In Chapter 10 we explore continuous distributions of data, in particular exponential and normal distributions, as a context for introduction of improper integrals and a more detailed investigation of the concept of limit. In particular, the distribution function for the standard normal distribution leads to the error function. In Chapter 11 we look at how a computer or calculator might be generating values of this and other more common transcendental functions. This leads to polynomial approximation and then to infinite series as convenient and very useful representations of functions.

Early versions of this text were distributed by the authors under the title of *The Calculus Reader*. Revised preliminary versions were published by D. C. Heath under the titles *The Calculus Reader* and *Project CALC: Chapters 1–6*. A companion volume for multivariable calculus is already being distributed by D. C. Heath in preliminary form. The multivariable material will be published in a separate volume.

Supplements

Instructor's Guide This manual explains how to use this text with a variety of approaches to calculus. We discuss:

- working with groups,
- assigning, responding to, and grading written work,
- getting students to read mathematics,
- integration of classroom and laboratory activities, and
- the use of gateway tests to check competency in symbol manipulation.

In addition, we give detailed suggestions for using the text on a week-by-week basis.

Complete Solutions Manual This manual contains complete solutions for all of the exercises in the text.

Laboratory Manuals The following manuals contain laboratory activities designed for a once-a-week lab that is closely correlated with the text material. These activities usually focus on interactive explorations and modeling of real-world problems with real data. They include problems from biology, physics, economics, epidemiology, and statistics.

- **TI-82/85 Laboratory Manual**
- **Mathematica Laboratory Manual**
- **Derive Laboratory Manual**
- **Maple Laboratory Manual**
- **HP-48 Laboratory Manual**
- **Mathcad for Windows Laboratory Manual**

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D. A. S.
L. C. M.

Introduction

What is this book about?

Most mathematics books are about answers — and how to get them. This book is about *questions* — and what to do about them. Like the world around us, this book has more questions than answers. Indeed, our questions *are* the questions of the world around us. Almost everything of importance in our world is moving or changing, and **calculus is the mathematical language of motion and change.**

To give you some idea of the importance of our subject, we will pose some questions that calculus might help us answer. You won't find the answers in the back of the book — indeed, answers that fit neatly in books are seldom real solutions to real problems. Here we go.

- Are we in the midst of a global population explosion? If so, what resources will we exhaust first: food, fuel, or terrestrial space? As a response to such a crisis, should we colonize outer space? If so, what would it take to do that, and how do we go about it? Can people survive in large numbers on the moon or Mars? Can we move enough of them there to make any difference? If so, what are the scientific, engineering, economic, political, sociological, theological, and biomedical problems we would have to solve? How do we solve them?

- Suppose we find there *is* a population crisis, but there is no viable solution to the problem of space colonization. What problems would we have to solve to continue our existence in relative peace on Earth? Population control? Waste management? Pollution control? Technological advances in computers, consumer goods, weapons, communications? Arms control or reduction? Management of international relations? Peace through strength or strength through peace? Economic growth or economic stability?

- Suppose there is *no* impending population explosion — population may be self-limiting. What then? Will we see world population level off at some stable number? If so, how big can we expect that number to be? Would its sheer size lead us to grapple with a host of other problems, such as extreme scarcity of resources and drastically lowered standards of living?

- If there is no leveling-off point, will there be oscillations in the population level? If so, will these be wild swings between very high and very low levels, or will they be modest variations at manageable levels? If the latter — which would suggest that population problems need not be high on our priority scale — what *are* the important problems of a society and a world that appear to be changing ever more rapidly?

These are challenging questions about *change*, more precisely about the *rates* at which dynamic quantities change and about the consequences we can determine from those rates. Calculus provides us with the conceptual framework and many of the computational tools for the quantitative and qualitative study of rates of change, and that's what this book is about.

Why study calculus?

We often ask our beginning students why they are taking calculus. Here are a few of the most common answers to this question.

- It's required for my major.
- I have always had a mathematics course, and this was the next one in line.
- My parents said I had to take it.
- I like mathematics.
- Everyone says mathematics is important — I just felt that I ought to do it.
- Calculus is central to understanding the development of philosophy and science in the last three centuries. Without a thorough grasp of this fundamental branch of mathematics, one cannot be considered an educated person.

Well, honesty compels us to admit that no one has actually given the last response, but our hope springs eternal.

Communication and cooperation

Often a problem comes to us in the form of data: An object falls through the air, and we observe data consisting of distances fallen at, say, ten different times. Can we tell how far the object had fallen at some time other than those at which we made the observations? Can we predict how a similar object will fall in the future? In this case, the theory comes to us from physics. The language of the theory is a mixture of English and calculus, and the calculations necessary to answer the questions require the same mix.

We concentrate on the use of calculus to solve problems. What has English — reading, writing, speaking — to do with solving problems? Problem solving requires deciding what should be done, executing the calculations, and interpreting the results. The environment for this intellectual activity is language, English in our case. Until you can describe what you have done, why you did it, and what it means, you have not solved the problem. For this reason we expect you to write up the projects on which you will work — in class or lab, or on your own time.

This textbook, your calculator or computer, and your instructor are all important resources for learning about calculus and the art of problem solving. There is another resource just as important as those already mentioned: your fellow students. We expect you to work on projects in teams, to talk about what you are doing, to explain your ideas and insights to each other. In the course of this work, you will find your fellow students an excellent source of help for understanding the course in general. Whatever your question, it is likely that somebody else in the class has considered it already and has some ideas for an answer. The key here is to talk to one another. When you do not understand why one thing follows from another, say so. When you do not see the evidence to support a conclusion, say so.

Learning is a cooperative — not a competitive — activity. We are about to embark on a great cooperative adventure: learning calculus. *Bon voyage!*

Contents

CHAPTER

1

Relationships

As students of calculus, nothing is more important to us than **functions.** *Other things may be more important to you in other contexts—for example, social justice, Mozart, or baseball. But this course is about functions—their various representations, their rates of growth and decay, and their uses in solving problems in many different disciplines. Thus, in this introductory chapter, we take up one big question:*

What is a function?

It's very important that we have a common understanding about the answer to this question. Otherwise, it will be difficult to make much sense of the rest of the course. Furthermore, our answer to this big question may or may not agree with your previous use of the word "function" in other mathematics courses. We will start with your prior understanding and work toward a new understanding that will serve us better for your study of calculus and its uses.

1.1 Related Variables

No doubt you already know an answer to the big question — What's a function? — from your previous study of mathematics. In fact, we will use your prior knowledge as our starting point for discussion and refinement of this central concept. Of course, only *you* can provide that prior knowledge.

Exploration Activity 1

Somewhere that you can keep track of it, write your present answer to the big question. You may use definitions, examples, symbols, or whatever you think appropriate for explaining to someone else what a function is.

The word "function," like all the other words we shall use in this course, belongs to the English language. As such, it has a definition — or several:

> **function** (fungk′sh*e*n) *n.* **1.** The action for which a person or thing is particularly fitted or employed. **2. a.** Assigned duty or activity. **b.** Specific occupation or role: *in his function as attorney*. **3.** An official ceremony or formal social occasion. **4.** Something closely related to another thing and dependent on it for its existence, value, or significance: *Growth is a function of nutrition.*[1]

Do you see any point of contact between that dictionary definition and what you wrote in response to Exploration Activity 1? Perhaps not — and therein lies a fundamental difficulty in the study of mathematics. Math books and math teachers seem to use words from the English language, but often with meanings that seem arbitrary and unrelated to common usage. Actually, as we shall see, definition **4.** is rather close to the meaning we shall establish in this chapter. If you wrote something like that, give yourself a pat on the back. If nothing you wrote looks like that, don't despair — by the end of this chapter we will have connected your prior experience with the concept of "function" as the word will be used in this course.

You may have noticed that the title of this chapter is not "Functions" but "Relationships." The key word in definition **4.** is "related," and we will establish our meaning for the word "function" within the more general context of relationships.

Variables and Data

We start with examples of quantitative relationships. Think for a moment about each of the following questions. You are not expected to know answers to these questions, just to be willing to think about them.

- The United States has a serious dropout problem. What is the relationship between state expenditures on teacher salaries and high school graduation rates?

- For adults, high blood pressure is linked to weight. Is there a similar relationship for children or adolescents?
- Will the world be seriously overpopulated in 20 years?

These questions are different in many respects, but answering each requires collection, organization, and interpretation of data. Each requires analysis of the relationship between *two variables*. In the first question, the two variables are the state expenditure on teacher salaries and the high school graduation rate. In the second question, the variables are blood pressure and weight. And in the third, the variables are time and population.

We can classify possible relationships between pairs of variables in four categories:

- One variable may have a causative effect on the other. For example, we expect that blood pressure in adults of the same height depends in some way on weight — higher weight causes higher blood pressure — but probably not the other way around.
- Other times there is a relationship between the variables, but it is not one of cause and effect. For example, at any given time, some number is the actual population of the world — but we would not consider either time or population to be a cause of the other.
- We might find that there is no relationship at all between the two variables. For example, we do not expect a relationship between the distance from a student's home to college and her height.
- Finally, we may not know if there is a relationship between two variables. For example, there might be some relationship between the annual dollar amount of imports from Mexico to the United States and the annual dollar amount of exports from the United States to Mexico — but, without any data, we have no way of knowing.

Checkpoint 1

For each of the following descriptions, state whether you think there is a relationship between the two variables and if either of them might have a causative effect on the other.

(a) Annual crop yield and annual rainfall in a given area.

(b) Temperature of an object measured in degrees Fahrenheit and temperature measured in degrees Celsius.

(c) The number of students entering Michigan State University in a given year and the year of entry.

(d) State expenditures on teachers' salaries and high school graduation rates.

To determine whether a relationship exists between two variables, we must analyze pairs of data — each pair consisting of a value of the first variable and a corresponding value of the second variable. Sometimes these data are gathered from a well-designed, carefully controlled scientific experiment, as might be the case for a study of blood pressure or crop yields. Other times we want to analyze data that already exist in the world around us, such as census data on populations.