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Wavelet Methods in Mathematical Analysis and Engineering

数学分析与工程中的小波方法

Alain Damlamian
Stéphane Jaffard
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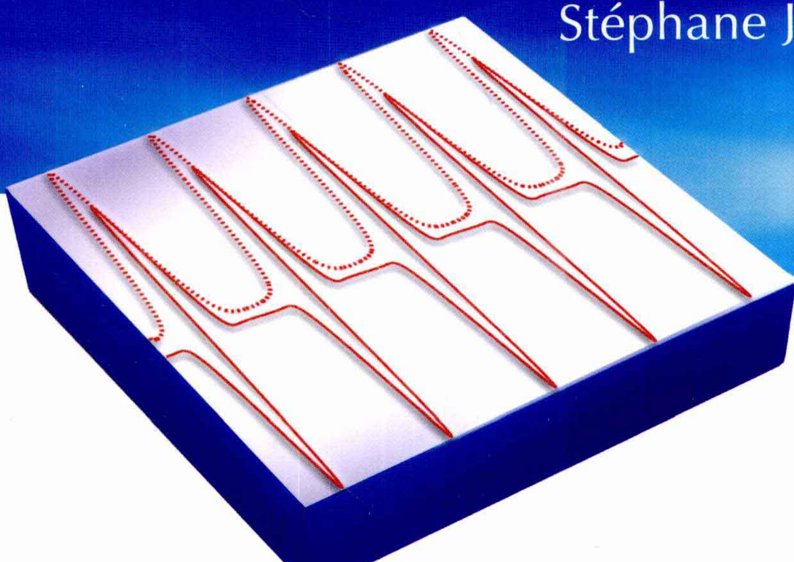

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Preface

The texts of this book grew from an ISFMA (Sino-French Institute of Applied Mathematics) symposium on "Wavelet Methods in Mathematical Analysis and Engineering" that took place in August 2007 on the Zhuhai campus of Sun Yat-Sen University. This symposium was composed of a one week summer school mainly directed towards Chinese PhD students and postdocs, followed by a one week conference attended by researchers from all over the world. This event was co-organized by Sun Yat-Sen University in Guangzhou and the ISFMA in Shangai. The purpose of the courses was to give the students the required level in wavelet analysis and in the main applications that would be treated during the conference, so that they would be able to follow it with profit. The courses were given by Albert Cohen, Daoqinq Dai, Stéphane Jaffard, Lixin Shen, Zuowei Shen and Lihua Yang; they covered basic materials *concerning construction of properties of wavelet bases*, and alternative decomposition methods; they gave an overview of the main applications in the numerical analysis of PDEs, and signal and image processing. The workshop exposed new techniques such as Empirical Mode Decomposition (EMD) and new trends in the recovery of missing data, such as compressed sensing, and a sample of a few recent key applications of wavelets in several scientific areas. This event was part of a long term collaboration between Chinese, Singaporean and French mathematicians in the area of wavelet analysis, and a second event took place one year later, with the "Chinese-French-Singaporean Joint Workshop on Wavelet Theory and Applications" in Singapore (June 2008).

These texts essentially correspond to the courses that were given during the summer school. Put together, they give a comprehensive overview of both the fundamentals of wavelet analysis and related tools, and of the most active directions of applications that developed recently. They offer a state of the art in several active areas of research where wavelet ideas, or more generally multiresolution ideas have proved particularly effective.

The paper by Jianfeng Cai, Raymond Chan, Lixin Shen and Zuowei Shen deals with the practical problems of high resolution reconstruction of noisy and blurred images, and the super-resolution challenge, which consists in using *a priori* information on the structure of the image in order to reconstruct it at a higher resolution than is available; this ill-posed problem is tackled using multiresolution ideas, which were at the very origin of wavelet techniques, in the 1980s. The resolution of these questions is obtained through the use of tight-framelets; the paper first gives a tutorial on frames and tight frames, which are now an important tool in signal and image processing, and then focuses on tight-framelets, a tool which is exposed in details, and whose efficiency is demonstrated in this context.

The paper by Albert Cohen addresses a fundamental problem in approximation theory: how to approximate a piecewise smooth function, in a numerically efficient way by few simple "building blocks". One key idea developed in the paper is that one should use nonlinear approximation techniques: one picks the approximation from a set of functions depending on N parameters (but which does not form a vector space), and the paper starts by a tutorial on N-term approximation. For the specific image processing problem which is proposed, the author developed a particularly effective method where the building blocks are piecewise polynomial functions on triangles on which no shape restriction is imposed. This extra flexibility has the advantage of offering efficient reconstruction algorithms for functions with edges along smooth lines. The algorithms used develop refinements techniques based on multiresolution ideas. The paper demonstrates the accuracy and the numerical simplicity of the method.

The paper by Stéphane Jaffard, Patrice Abry, Stéphane G. Roux, Béatrice Vedel and Herwig Wendt gives a brief tutorial on constructions of wavelet bases, and the characterization of function spaces in terms of wavelet coefficients. It shows the relevance of these techniques in a problem posed by the seminal papers of Kolmogorov in turbulence in the 1940s, where he advocated the study of some quantities which were expected to be scaling invariant, and fundamental for the comprehension of small scale turbulence. The study of these quantities rewritten through a wavelet expansion yields unexpected new tools for signal and image classification and for the selection of turbulence models. Applications to multifractal analysis are given, i.e. for the estimation on the size of the sets of points where a function has a given pointwise Hölder regularity.

The paper by Chaochun Liu and Daoqing Dai addresses the question of face recognition. The difficulty of the problem arises form the fact that a face can change widely due to variations in pose, expression and illumination. The challenge is to find attributes that remain stable under such variations. This is precisely supplied by wavelet techniques, which yield an efficient tool for feature extraction; this property is in agreement with the important discovery that the human visual system indeed uses a kind of wavelet decomposition (Gabor wavelets that are based on a space-frequency decomposition) as a preprocessing tool, in particular for recognition. Furthermore, wavelets are simultaneously useful in this context for denoising. The paper first proposes a tutorial on the problem of face recognition, and then focuses on wavelet-based algorithms.

The paper by Lihua Yang gives an introduction to the Hilbert-Huang transform. It supplies its theoretical mathematical background and shows recent applications to pattern recognition. A key (but ill-posed) problem in signal processing is to define the instantaneous frequency of a signal. A classical way to define this notion is to use the Hilbert transform and the associated analytic signal; however, if applied directly to a complex signal, this method can lead to severe instabilities and ill-functionings. A fundamental advance was obtained with the introduction the Empirical Mode Decomposition, which splits the signal into simpler basic components: the Intrinsic Mode Functions; it can be efficiently used as a preprocessing, since the instantaneous frequency of each simple component can then be determined in a numerically meaningful and stable way. This paper shows recent applications of the Hilbert-Huang transform e.g. to a tsunami wave, and to pattern recognition.

Alain Damlamian, Stéphane Jaffard

Editors

May 2010

Contents

Tight Frame Based Method for High-Resolution Image Reconstruction

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Abstract

We give a comprehensive discussion on high-resolution image reconstruction based on a tight frame. We first present the tight frame filters arising from the problem of high-resolution image reconstruction and the associated matrix representation of the filters for various boundary extensions. We then propose three algorithms for high-resolution image reconstruction using the designed tight frame filters and show analytically the properties of

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these algorithms. Finally, we numerically illustrate the efficiency of the proposed algorithms for natural images.

1 High-resolution image reconstruction model

The problem of high-resolution image reconstruction is to reconstruct a high-resolution (HR) image from multiple, under-sampled, shifted, degraded and noisy frames where each frame differs from the others by some sub-pixel shifts. The problem arises in a variety of scientific, medical, and engineering applications. The problem of HR image reconstruction is a hot field. In the past few years, two special issues on the topic was published: IEEE Signal Processing Magazine (Volume 20, Issue 3, May 2003) and International Journal of Imaging Systems and Technology (Volume 14, No. 2, 2004).

The earliest study of HR image reconstruction was motivated by the need to improve the resolution of images from Landsat image data. In [28], Huang and Tsay used the frequency domain approach to demonstrate the improved reconstruction image from several down-sampled noise-free images. Later on, Kim *et al.* [30] generalized this idea to noisy and blurred images. Both methods in [28, 30] are computational efficiency, but they are prone to model errors, and that limits their use [1]. Statistical methods have appeared recently for super-resolution image reconstruction problems. In this direction, tools such as a maximum a posteriori (MAP) estimator with the Huber-Markov random field prior and a Gibbs image prior are proposed in [25, 43]. In particular, the task of simultaneous image registration and super-resolution image reconstruction are studied in [25, 45]. Iterative spatial domain methods are one popular class of methods for solving the problems of resolution enhancement [3, 21, 22, 23, 27, 31, 32, 36, 38, 39, 41]. The problems are formulated as Tikhonov regularization. A great deal of work has been devoted to the efficient calculation of the reconstruction and the estimation of the associated hyperparameters by taking advantage of the inherent structures in the HR system matrix. Bose and Boo [3] used a block semi-circulant matrix decomposition in order to calculate the MAP reconstruction. Ng *et al.* [36] and Ng and Yip [37] proposed a fast discrete cosine transform based approach for HR image reconstruction with Neumann boundary condition. Nguyen *et al.* [40, 41] also addressed the problem of efficient calculation. The proper choice of the regularization tuning parameter is crucial to achieving robustness in the presence of noise and avoiding trial-and-error in the selection of an optimal tuning parameter. To this end, Bose *et al.* [4] used an L-curve based approach. Nguyen *et al.* [41] used a generalized cross-validation method. Molina *et al.* [33] used an expectation-maximization algorithm. Lu *et al.* [32] proposed multiparameter regularization methods which introduce different regularization parameters for different frequency bands of the regularization operator.

Low-resolution images can be viewed as outputs of the original high-resolution image passing through a low-pass filter followed by a decimation process. This viewpoint suggests that a framework of multiresolution analysis can be naturally adopted to produce an HR image from a set of low-resolution images of the same scene with sub-pixel shifts. In this fashion, a series of work has been done recently, see, e.g., [9, 10, 11, 12, 13]. Extension of these work will be discussed in the paper.

Here we present a mathematical model proposed by Bose and Boo in [3] for high-resolution image reconstruction. Consider $K \times K$ sub-window-shifted low-resolution images in which each image has $N_1 \times N_2$ interrogation windows and the size of each interrogation window is $T_1 \times T_2$. Here, $K \times K$ denotes K shifts in both the vertical and horizontal directions. The goal is to reconstruct a much higher resolution image with $M_1 \times M_2$ sub-windows, where $M_1 = K \times N_1$ and $M_2 = K \times N_2$.

In order to have enough information to resolve the high-resolution image, it is assumed that there are sub-window shifts between the low-resolution images. For a low-resolution image denoted by (k_1, k_2), where $0 \leq k_1, k_2 < K$ with $(k_1, k_2) \neq (0, 0)$, its vertical and horizontal shifts $d^x_{k_1,k_2}$ and $d^y_{k_1,k_2}$ with respect to the $(0,0)$th reference low-resolution image are given by $d^x_{k_1,k_2} = \left(k_1 + \epsilon^x_{k_1,k_2}\right)\frac{T_1}{K}$ and $d^y_{k_1,k_2} = \left(k_2 + \epsilon^y_{k_1,k_2}\right)\frac{T_2}{K}$. Here $\epsilon^x_{k_1,k_2}$ and $\epsilon^y_{k_1,k_2}$ are the vertical and horizontal *shift errors* respectively. We assume that $|\epsilon^x_{k_1,k_2}| < \frac{1}{2}$ and $|\epsilon^y_{k_1,k_2}| < \frac{1}{2}$. Figure 1.1 shows the example of 2×2 shifted low-resolution images.

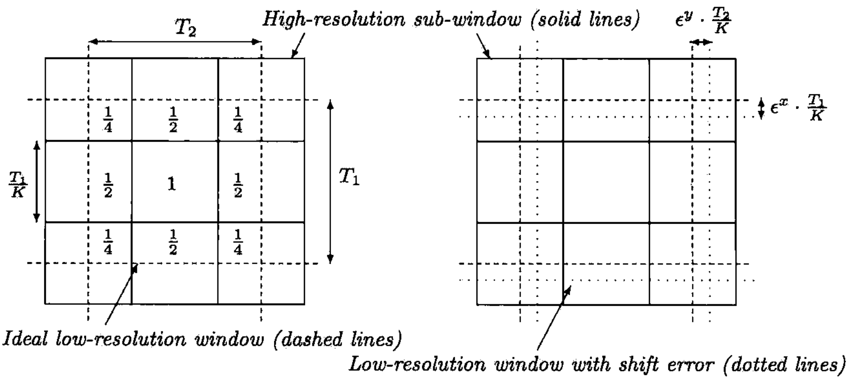

Figure 1.1 Windows without and with shift error when $K = 2$ (left and right respectively).

For a low-resolution image (k_1, k_2), the average quantity at its (n_1, n_2)th interrogation window is modelled by:

$$g_{k_1,k_2}[n_1,n_2] = \frac{1}{T_1 T_2} \int_{A_{k_1,k_2;n_1,n_2}} f(x,y)\, dx\, dy + \eta_{k_1,k_2}[n_1,n_2], \quad (1.1)$$

where the interrogation window in the low-resolution image is

$$A_{k_1,k_2;n_1,n_2} = \left[T_1\left(n_1 - \frac{1}{2}\right) + d^x_{k_1,k_2}, T_1\left(n_1 + \frac{1}{2}\right) + d^x_{k_1,k_2}\right] \times \left[T_2\left(n_2 - \frac{1}{2}\right) + d^y_{k_1,k_2}, T_2\left(n_2 + \frac{1}{2}\right) + d^y_{k_1,k_2}\right].$$

Here (n_1, n_2) indicates an interrogation window in the low-resolution image (k_1, k_2) (where $0 \le n_1 < N_1$ and $0 \le n_2 < N_2$) and $\eta_{k_1,k_2}[n_1,n_2]$ is the noise (refer to [3]). We interlace all the sub-window-shifted low-resolution images g_{k_1,k_2} to form an $M_1 \times M_2$ image g by assigning

$$g[Kn_1 + k_1, Kn_2 + k_2] = g_{k_1,k_2}[n_1,n_2].$$

The pseudo high-resolution image g is called the *observed high-resolution image*.

The integral values on the sub-window of the high-resolution image is approximated by

$$f[i,j] = \frac{K^2}{T_1 T_2} \int_{A_{i,j}} f(x,y)\, dx\, dy, \quad 0 \le i < M_1, 0 \le j < M_2, \quad (1.2)$$

which is the average quantity inside the (i,j)th high-resolution sub-window:

$$A_{i,j} = \left[i\frac{T_1}{K}, (i+1)\frac{T_1}{K}\right] \times \left[j\frac{T_2}{K}, (j+1)\frac{T_2}{K}\right], \quad 0 \le i < M_1, 0 \le j < M_2. \quad (1.3)$$

To obtain the true high-resolution image f from the observed high-resolution image g, one will have to solve (1.1) for f. By discretizing (1.1) and (1.2) using the rectangular quadrature rule, we have

$$g_{k_1,k_2}[n_1,n_2] = \sum_{p,q=0}^{K} W[p,q] f[Kn_1+k_1+p, Kn_2+k_2+q] + \eta_{k_1,k_2}[n_1,n_2], \quad (1.4)$$

where the weighting matrix W for discretizing the integral equation (1.1) in the case without shift error is

$$W = \frac{1}{K^2} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \tag{1.5}$$

which is assigned for associated sub-windows of the high-resolution image. Equation (1.4) is a system of linear equations relating the unknown values $f[i,j]$ to the given observed high-resolution image values $g[i,j]$.

For simplifying the exposition, f and g will be considered as the column vectors formed by $f[i,j]$ and $g[i,j]$. This linear system corresponding to (1.4) for high-resolution image reconstruction is reduced to

$$Hf + \eta = g, \tag{1.6}$$

where the blurring matrix H, which is formulated from (1.4), varies under different boundary conditions and η is the noise vector. For the case without shift error, the blurring matrix H is given by

$$H = \frac{1}{K^4} \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \otimes \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where the Kronecker operator $\otimes$ is defined by $A \otimes B = [a_{ij}B]$ with $A = [a_{ij}]$. The key problem is to recover the true high-resolution image f from the observed high-resolution image g by solving (1.6).

If the low-resolution images are shifted by exactly half of the window, then the problem reduces to solving a spatially invariant linear system. Depending on the boundary conditions we impose on the images, the coefficient matrix H is either Topelitz or Toeplitz-like. The model was then solved in [3, 36] using preconditioned conjugate gradient method.

We next discuss in details several approaches that will use the tight frame for solving the system (1.4) or (1.6). The performance of these methods will be examined in numerical simulations. In the next section, we will give a brief review on the frame theory. In particular, we will present the tight frame system with (1.5) as its low-pass filter.

The outline of this paper is as follows. In Section 2, we give a brief review on tight frames with an emphasis on the unitary extension principle. Section 3 contains four main parts. The first part presents the tight frames arising from the problem of HR image reconstruction. The matrix representations of the tight frame filters associated with the HR image reconstruction are given, by imposing the periodic and symmetric boundary conditions, in the second and third parts, respectively. It follows by showing the multi-level framelet decomposition and reconstruction in the last part. We propose three framelet-based algorithms to tackle the problem of HR image reconstruction in Section 4. In particular, we give a complete analysis for Algorithm I in Section 5. Numerical experiments for all three algorithms are presented in Section 6.

2 Preliminaries on tight framelets

The notion of frame was first introduced by Duffin and Schaeffer [20] in 1952. A countable system $X \subset L^2(\mathbb{R})$ is called a *frame* of $L^2(\mathbb{R})$ if

$$\alpha\|f\|_2^2 \le \sum_{h\in X} |\langle f, h\rangle|^2 \le \beta\|f\|_2^2, \tag{2.1}$$

where the constants α and β, $0 < \alpha \le \beta < \infty$, are lower and upper bounds of the frame system X. The notation $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2 = \langle\cdot,\cdot\rangle^{1/2}$ are the inner product and norm of $L^2(\mathbb{R})$. When $\alpha = \beta (= 1)$, the frame system X is called a *tight frame*. In what follows, our discussion is concentrated on the tight frame.

Two operators, namely analysis operator and synthesis operator, are associated with the tight frame. The analysis operator of the frame is defined as

$$\mathcal{F} : L^2(\mathbb{R}) \longrightarrow \ell^2$$

with

$$\mathcal{F}(f) = \{\langle f, h\rangle\}_{h\in X}.$$

Its adjoint operator $\mathcal{F}^*$, called the synthesis operator, is defined as

$$\mathcal{F}^* : \ell^2 \longrightarrow L^2(\mathbb{R})$$

with

$$\mathcal{F}^*(c) = \sum_{h\in X} c_h h, \quad c = \{c_h\}_{h\in X}.$$

Hence, X is a tight frame if and only if $\mathcal{F}^*\mathcal{F} = \mathcal{I}$. This is true if

$$f = \sum_{h\in X} \langle f, h\rangle h, \quad \forall f \in L^2(\mathbb{R}), \tag{2.2}$$

which is equivalent to

$$\|f\|_2^2 = \sum_{h \in X} |\langle f, h\rangle|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{2.3}$$

Equation (2.2) is the perfect reconstruction formula of the tight frame. Identities (2.2) and (2.3) hold for an arbitrary orthonormal basis of $L^2(\mathbb{R})$. In this sense, an orthonormal basis is a tight frame, and a tight frame is a generalization of orthonormal basis. But tight frames sacrifice the orthonormality and the linear independence of the system in order to get more flexibility. Therefore tight frames can be redundant.

For a tight frame system X, we have

$$\sum_{h \in X} |\langle f, h\rangle|^2 \leq \sum_{h \in X} |c_h|^2$$

for all possible representation of $f = \sum_{h \in X} c_h h$, $\{c_h\} \in \ell^2$. In other words, the sequence $\mathcal{F}(f)$ obtained by the analysis operator $\mathcal{F}$ has the smallest ℓ^2 norm among all sequences $\{c_h\} \in \ell^2$ satisfying $f = \sum_{h \in X} c_h h$.

If $X(\Psi)$ is the collection of the dilations and the shifts of a finite set $\Psi \subset L^2(\mathbb{R})$, i.e.,

$$X(\Psi) = \{K^{i/2}\psi(K^i x - j) : \psi \in \Psi, i, j \in \mathbb{Z}\},$$

then $X(\Psi)$ is called a *wavelet* (or *affine*) *system of dilation* K. In this case the elements in Ψ are called the *generators.* When $X(\Psi)$ is a tight frame for $L^2(\mathbb{R})$, then $\psi \in \Psi$ are called (*tight*) *framelets.*

A normal framelet construction starts with a refinable function. A compactly supported function $\phi \in L^2(\mathbb{R})$ is *refinable* (a scaling function) with a refinement mask τ_ϕ if it satisfies

$$\widehat{\phi}(K\cdot) = \tau_\phi \widehat{\phi}.$$

Here $\widehat{\phi}$ is the Fourier transform of ϕ, and τ_ϕ is a trigonometric polynomial with $\tau_\phi(0) = 1$, i.e., a refinement mask of a refinable function must be a lowpass filter. One can define a multiresolution analysis from a given refinable function, details about that is omitted here, but can be found, for instance, in [19, 29].

For a given compactly supported refinable function, the construction of tight framelet systems is to find a finite set Ψ that can be represented in the Fourier domain as

$$\widehat{\psi}(K\cdot) = \tau_\psi \widehat{\phi}$$

for some 2π-periodic τ_ψ. The unitary extension principle (UEP) of [42] says that the wavelet system $X(\Psi)$ generated by a finite set Ψ forms a tight frame in $L^2(\mathbb{R})$ provided that the masks τ_ϕ and $\{\tau_\psi\}_{\psi\in\Psi}$ satisfy:

$$\tau_\phi(\omega)\overline{\tau_\phi\left(\omega+\frac{2\gamma\pi}{K}\right)}+\sum_{\psi\in\Psi}\tau_\psi(\omega)\overline{\tau_\psi\left(\omega+\frac{2\gamma\pi}{K}\right)}=\delta_{\gamma,0},$$
$$\gamma=0,1,\cdots,K-1 \tag{2.4}$$

for almost all ω in $\mathbb{R}$. Practically, we require all masks to be trigonometric polynomials. Thus, (2.4) together with the fact that $\tau_\phi(0)=1$ imply that $\tau_\psi(0)=0$ for all $\psi\in\Psi$. Hence, $\{\tau_\psi\}_{\psi\in\Psi}$ must correspond to high-pass filters. The sequences of Fourier coefficients of τ_ψ, as well as τ_ψ itself, are called *framelet masks*. The construction of framelets Ψ essentially is to design, for a given refinement mask τ_ϕ, framelet masks $\{\tau_\psi\}_{\psi\in\Psi}$ such that (2.4) holds. A more general principle of construction tight framelets, the oblique extension principle, was developed recently in [14, 17].

In the next section, we will use the EUP to construct a framelet system arising from the problem of HR image reconstruction.

3 Tight frame system arising from high-resolution image reconstruction

3.1 Filter design

The low-pass filter (1.5) for high-resolution image reconstruction is separable and can be written as follows

$$W=h_0^T h_0,$$

where

$$h_0=\frac{1}{K}\left[\frac{1}{2},1,\cdots,1,\frac{1}{2}\right].$$

Hence, to design a tight frame system with W as its low-pass filter, we just need to construct a tight frame system with h_0 as its low-pass filter. By virtue of the Fourier series of h_0, define

$$\widehat{\phi}(\omega):=\prod_{j=1}^{\infty}\widehat{h}_0(K^{-j}\omega), \tag{3.1}$$

where

$$\widehat{h}_0(\omega)=\frac{1}{2K}+\frac{1}{K}\left(\sum_{k=1}^{K-1}\exp(-ik\omega)\right)+\frac{1}{2K}\exp(-iK\omega).$$

It was shown in [44] that ϕ is in $L^2(\mathbb{R})$ and Hölder continuous with Hölder exponent $\ln 2/\ln K$.

For any integer $L \geq 2$, define

$$m_{L,p} := \frac{\sqrt{2}}{L}\left[\cos\left(\frac{p\pi}{2L}\right), \cos\left(\frac{3p\pi}{2L}\right), \cdots, \cos\left(\frac{(2L-1)p\pi}{2L}\right)\right],$$

and their Fourier series

$$\widehat{m}_{L,p}(\omega) = \frac{\sqrt{2}}{L}\sum_{\ell=1}^{L}\cos\left(\frac{(2\ell-1)p\pi}{2L}\right)\exp(-i\ell\omega),$$

for $p = 1, \cdots, L-1$. We further define, for any integer $K \geq 2$,

$$\widehat{h}_{2p+q}(\omega) := \widehat{m}_{2,q}(\omega)\widehat{m}_{K,p}(\omega), \tag{3.2}$$

where $p = 1, \cdots, K-1$, $q \in \{0, 1\}$. We can easily check that

$$\sum_{q=0}^{1}\sum_{p=0}^{K-1}\widehat{h}_{2p+q}(\omega)\overline{\widehat{h}_{2p+q}(\omega + \frac{2\pi\ell}{K})} = \delta_{\ell,0}, \quad \ell = 0, 1, \cdots, K-1. \tag{3.3}$$

The EUP of [42] yields that the functions

$$\Psi = \{\psi_{2p+q} : 0 \leq p \leq K-1, \quad q = 0, 1, \quad (p,q) \neq (0,0)\}$$

defined by

$$\widehat{\psi}_{2p+q}(\omega) = \widehat{h}_{2p+q}\left(\frac{\omega}{K}\right)\widehat{\phi}\left(\frac{\omega}{K}\right)$$

are tight framelets. Furthermore,

$$X(\Psi) = \Big\{K^{k/2}\psi_{2p+q}(K^k \cdot -j) : 0 \leq p \leq K-1,$$
$$q = 0, 1, (p,q) \neq (0,0); k, j \in \mathbb{Z}\Big\}$$

is a tight frame system of $L^2(\mathbb{R})$.

In the following discussion, we always assume that the indexes of all filters h_ℓ, run from $-K/2$ to $K/2$ for even number K and $-(K+1)/2$ to $(K-1)/2$ for odd number.

We are interested in the matrix representation of the identity

$$\sum_{q=0}^{1}\sum_{p=0}^{K-1}|\widehat{h}_{2p+q}(\omega)|^2 = 0 \tag{3.4}$$

for filters given by (3.2). In image processing, periodic and symmetric boundary conditions are usually imposed to give matrix representation of (3.4). In the following subsections, we will give the corresponding representations for both boundary conditions.

3.2 Matrix representation of filters with periodic boundary conditions

For simplicity, we are not going to write the matrix forms of the filters given by (3.2) for a general integer K. Instead, we give these matrices for the filters with $K = 2$ and $K = 3$ only. From there, one can easily give the matrix representation for filters associated with any integer K.

Example 1. *For $K = 2$, we have, from (3.2), the low-pass filter $h_0 = \frac{1}{4}[1, 2, 1]$ and three high-pass filters $h_1 = \frac{1}{4}[1, 0, -1]$, $h_2 = \frac{1}{4}[1, 0, -1]$, and $h_3 = \frac{1}{4}[1, -2, 1]$, respectively. The corresponding matrix representation under the periodic boundary condition for filters h_0, h_1, h_2, and h_3 are circulant matrices H_0, H_1, H_2, and H_3, respectively, with their first rows being the following*

$$\left[\frac{1}{2}, \frac{1}{4}, 0, \cdots, 0, \frac{1}{4}\right], \quad \left[0, -\frac{1}{4}, 0, \cdots, 0, \frac{1}{4}\right],$$
$$\left[0, -\frac{1}{4}, 0, \cdots, 0, \frac{1}{4}\right], \quad \left[-\frac{1}{2}, \frac{1}{4}, 0, \cdots, 0, \frac{1}{4}\right].$$

We can check that

$$H_0^T H_0 + H_1^T H_1 + H_2^T H_2 + H_3^T H_3 = I.$$

We remark that $h_1 = h_2$ in above tight frame filters. We can merge h_1 and h_2 together and deduce a new tight frame system with the low-pass filter $\frac{1}{4}[1, 2, 1]$ and two high-pass filters $\frac{\sqrt{2}}{4}[1, 0, -1]$ and $\frac{1}{4}[1, -2, 1]$. A similar situation happens in the next example.

Example 2. *For $K = 3$, we have the low-pass filter $h_0 = \frac{1}{6}[1, 2, 2, 1]$ and five high-pass filters $h_1 = \frac{1}{6}[1, 0, 0, -1]$, $h_2 = \frac{\sqrt{6}}{12}[1, 1, -1, -1]$, $h_3 = \frac{\sqrt{6}}{12}[1, -1, -1, 1]$, $h_4 = \frac{\sqrt{2}}{12}[1, -1, -1, 1]$, and $h_5 = \frac{\sqrt{2}}{12}[1, -3, 3, -1]$. The corresponding matrix representation under the periodic boundary condition for filters $h_0, h_1, \cdots, h_5$ are circulant matrices $H_0, H_1, \cdots, H_5$, respectively, with their first rows being the following*

$$\frac{1}{6}[2, 1, 0, \cdots, 0, 1, 2], \quad \frac{1}{6}[0, -1, 0, \cdots, 0, 1, 0],$$
$$\frac{\sqrt{6}}{12}[-1, -1, 0, \cdots, 0, 1, 1], \frac{\sqrt{6}}{12}[-1, 1, 0, \cdots, 0, 1, -1],$$
$$\frac{\sqrt{2}}{12}[-1, 1, 0, \cdots, 0, 1, -1], \quad \frac{\sqrt{2}}{12}[3, -1, 0, \cdots, 0, 1, -3].$$

Again, it can be easily checked that

$$H_0^T H_0 + H_1^T H_1 + H_2^T H_2 + H_3^T H_3 + H_4^T H_4 + H_5^T H_5 = I.$$