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Miroslav Pavlović

FUNCTION CLASSES ON THE UNIT DISC

AN INTRODUCTION

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To my family

Preface

This is an attempt to write a book that differs as much as possible from the existing[1] books in this area. Although the main protagonists of the story, *Hardy, Bergman, Besov, Lipschitz, Bloch, Hardy–Sobolev*, BMO, etc., are well known through many books, some new properties of them have been described, whereas verifications of known properties are in many cases new. The reader is assumed to be well acquainted with complex analysis and the theory of Lebesgue integration, which includes the fundamental facts of the harmonic functions theory – Fatou's theorem on radial limits of the Poisson integral of a complex Borel measure, along with the canonical isometry between the harmonic Hardy space h^p and the Lebesgue space L^p ($p > 1$). The knowledge of a minimum of the theory of Fourier series and Banach space techniques is also desirable. All this, and much more, can be found in Rudin's *Real and complex analysis*.

Some deep facts on Lebesgue spaces and maximal functions stated without proofs in Appendix B, e.g. the Fefferman–Stein vector maximal theorem and a theorem of Nikishin, only should be understood and taken as granted. One more fact of such deepness is used in Chapter 5, and concerns the real interpolation between Hardy spaces, but it arises because of the author's ineffectiveness to find a simple proof, which certainly exists, of a theorem on radial limits of "Hardy–Bloch" functions. The author hopes that applications of these theorems in this text shows their strength and that this can motivate the reader to learn the corresponding theories.

The exposition is not linear but the reader can be sure that there are no circular arguments in the text.

Approximately 30 percent of the text already appeared in the author's booklet *Introduction to Function Spaces on the Disk* [374],but "Classes" cannot be treated as an expanded version of "Spaces" because the latter is not a subset of the former, and the organization of text is significantly different.

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of Illinois at Urbana-Champaign), and Ern Gun Kwon (Andong National University), who provided me with texts that were unavailable in Belgrade; and, especially, an anonymous reviewer whose suggestions significantly affected the organization of the text.

Belgrade, July 2013 Miroslav Pavlović

Contents

1 The Poisson integral and Hardy spaces

This chapter contains the basic properties of the Poisson integral of an L^1-function and, more generally, of a complex measure on the circle $\mathbb{T}$. Fatou's theorem on radial limits, the Privalov–Plessner on the radial limits of the conjugate function, the Fefferman–Stein theorem on subharmonic behavior of $|f|^p$, and the Riesz projection theorem are some of the most important results of the chapter. Also, the well-known connection between the harmonic Hardy space h^p ($1 \leq p \leq \infty$) and the Lebesgue space $L^p(\mathbb{T})$ is presented without proof. A brief discussion of h^p for $p < 1$ is in Section 1.3. In the last section we present a quick introduction to basic properties of (analytic) Hardy spaces. Our approach differs from that in other texts [129, 159, 273, 425, 430, 525] in which we first prove the Hardy–Littlewood decomposition lemma, and then deduce the radial limits theorem, and some other fundamental results due to F. and M. Riesz, Smirnov, Szegö, Kolmogorov et al., without using Blaschke products. At one place we use the Hardy–Littlewood complex maximal theorem although we consider the maximal functions in Appendix B, Section B.3. However, the reader can treat Section B.3 as a part of this chapter inserted before considering Hardy spaces.

Preliminaries

Some notation

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ the real line, the complex plane, the set of all integers, and the set of nonnegative integers, respectively. By $\mathbb{R}_+$ and $\mathbb{N}_+$ we denote the set of positive real numbers and the set of positive integers. If $d\mu$ is a finite positive measure on a sigma-algebra of subsets of a set S, we write

$$\fint_S f\, d\mu = \frac{1}{\mu(S)} \int_S f\, d\mu,$$

and in particular

$$\fint_0^{2\pi} f(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta, \quad \fint_{\mathbb{D}} f\, dA = \frac{1}{\pi} \int_{\mathbb{D}} f\, dA,$$

where dA is the Lebesgue measure on $\mathbb{C}$ and $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$. Similarly

$$\fint_{\mathbb{T}} f(\zeta)\, |d\zeta| = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta)\, |d\zeta|, \quad \text{where } \mathbb{T} = \partial\mathbb{D}.$$

The arc-length measure on $\mathbb{T}$ will be denoted by dl and so

$$\int_{\mathbb{T}} f(\zeta)\, |d\zeta| = \int_{\mathbb{T}} f\, dl.$$

The two-dimensional measure of a measurable set $G \subset \mathbb{C}$ will be denoted by $|G|$. Similarly, $|S|$ denotes the arc-length measure of $S \subset \mathbb{T}$.

When dealing with spaces of analytic or harmonic functions it is convenient to use a new symbol, "$\diamond$", that has the following properties:

$$\frac{1}{\diamond} = 0 \quad \text{and} \quad x < \diamond < \infty \quad \text{for all } x \in \mathbb{R}.$$

Let $f \in L^p(\mathbb{T})$, $0 < p \leq \infty$, and

$$\|f\|_{L^p(\mathbb{T})} = \|f\|_p = \left(\fint_{\mathbb{T}} |f(\zeta)|^p \, |d\zeta| \right)^{1/p}.$$

We write $L^\diamond(\mathbb{T}) = C(\mathbb{T})$, and interpret the integral as in the case $p = \infty : \|f\|_\diamond = \|f\|_\infty = \|f\|_{L^\infty(\mathbb{T})}$. So we have $L^p(\mathbb{T}) \supsetneq L^q(\mathbb{T})$ for $0 < p < q \leq \infty$.

Möbius transformations of the unit disk

Every biholomorphic mapping (Möbius transformation) φ from $\mathbb{D}$ onto $\mathbb{D}$ can be represented as $\varphi(z) = b\sigma_a(z)$, where $|b| = 1$ and

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad |a| < 1, \ |z| \leq 1.$$

These transformations form a group, called the Möbius group and denoted by Möb($\mathbb{D}$), with respect to composition of mappings.

The functions σ_a have important properties:

- $\sigma_a^{-1} = \sigma_a$, where σ_a^{-1} denotes the inverse mapping.
- $\sigma_a'(z) := (\sigma_a)'(z) = -\frac{1-|a|^2}{|1-\bar{a}z|^2}$, $\quad |a| < 1, \ |z| \leq 1$.
- We have
$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} = |\sigma_a'(z)|(1 - |z|^2)$$
  and, more generally,
$$1 - \sigma_a(z)\overline{\sigma_a(w)} = \frac{(1 - |a|^2)(1 - z\bar{w})}{(1 - z\bar{a})(1 - a\bar{w})}.$$
- The functional $\eth(a, z) = |\sigma_a(z)|$ $(a, z \in \mathbb{D})$ is a metric on $\mathbb{D}$, and is called the pseudohyperbolic metric. It is Möbius invariant in sense that $\eth(\sigma(w), \sigma(z)) = \eth(w, z)$ for all $\sigma \in$ Möb($\mathbb{D}$) and $z, w \in \mathbb{D}$.
- The measure $d\tau(z) = (1 - |z|^2)^{-2} \, dA(z)$ is Möbius invariant, which means in particular that
$$\int_{\mathbb{D}} h \circ \sigma_a \, d\tau = \int_{\mathbb{D}} h \, d\tau,$$
  where $h \geq 0$ is a measurable function on $\mathbb{D}$.