Generalized Functions, Volume 4

Applications of Harmonic Analysis

I. M. Gel′fand
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Translator's Note

This translation differs from the Russian edition of 1961 mainly in that the authors have revised the proof contained in Section 3.2 of Chapter 1, as well as a major portion of Section 3.6 of the same chapter (in particular, the proof of Theorem 7). The symbols ▼ and ▲ in the margin indicate the beginning and end of a few passages in which I have ventured to deviate (in a not completely trivial way) from the original, for the purpose of eliminating a gap or an obscurity, and in one case (in Section 5.1, Chapter 4) of reducing a ten-line proof to the obvious one-line proof. In addition, I have allowed myself the luxury of a number of remarks, which appear in footnotes marked by a dagger (†). Any comments regarding these or other portions of this volume will be welcome.

June, 1964 AMIEL FEINSTEIN

Foreword

This book is the fourth volume of a series of monographs on functional analysis appearing under the title "Generalized Functions." It should not, however, be considered a direct sequel to the preceding volumes. In writing this volume the authors have striven for the maximum independence from the preceding volumes. Only that material which is discussed in the first two chapters of Volume 1 must be considered as the indispensable minimum which the reader is required to know. In view of this, certain topics which were discussed in the preceding volumes are briefly repeated here.

This book is devoted to two general topics: recent developments in the theory of linear topological spaces and the construction of harmonic analysis in n-dimensional Euclidean and infinite-dimensional spaces.

After the appearance of a theory of topological spaces, the question arose of distinguishing a class of topological spaces, defined by rather simple axioms and including all (or nearly all) spaces which arise in applications. In the same way, after a theory of linear topological spaces was created, it became necessary to ascertain which class of spaces is most suitable for use in mathematical analysis. Such a class of linear topological spaces—nuclear spaces—was singled out by the French mathematician A. Grothendieck.

The class of nuclear spaces includes all or nearly all linear topological spaces which are presently used in analysis, and has a number of extremely important properties: the kernel theorem of L. Schwartz is valid in nuclear spaces, as is also the theorem on the spectral resolution of a self-adjoint operator. Furthermore, any measure on the cylinder sets in the conjugate space of a nuclear space is countably additive. The first and fourth chapters of this book are devoted to the discussion of these questions. In connection with spectral analysis, the concept of a rigged Hilbert space is introduced, which turns out, apparently, to be very useful also in many other questions in mathematics.

The second question which we study in this volume is the harmonic analysis of functions in various spaces. Harmonic analysis in Euclidean space (the Fourier integral) has already been discussed to some extent in previous volumes. We have given up the idea of repeating here the material in the preceding volumes which was devoted to the Fourier integral (possibly, had all of the volumes been written at the same time, many questions in the theory of the Fourier integral, for example the Paley–Wiener theorem for generalized functions, would have found their natural setting in this volume). We discuss here only questions of harmonic analysis in Euclidean space which were left unclarified in the previous volumes. Namely, we consider the Fourier transformation of measures having one or another order of growth (the theory of generalized positive definite functions) and its application in the theory of generalized random processes. The Fourier transformation of measures in linear topological spaces is considered at the same time.

In the following, fifth volume, we single out questions of harmonic analysis on homogeneous spaces (in particular, harmonic analysis on groups) and intimately related questions of integral geometry on certain spaces of constant curvature. This theory, which is very rich in the diversity of its results (connected, for example, with the theory of special functions, analytic functions of several complex variables, etc.) could not, of course, be discussed in its entirety within the confines of the fifth volume. We have restricted ourselves to discussing only questions of harmonic analysis on the Lorentz group. It should be remarked that harmonic analysis on the Lorentz group and the related homogeneous spaces is a considerably richer subject than harmonic analysis in the "degenerate" case of a Euclidean space. For example, in the case of a Euclidean space only the smoothness of the Fourier transform of a function is influenced by specifying one kind of behavior or another at infinity of the function itself. But in the case of the Lorentz group, specifying the behavior of the function at infinity leads to certain algebraic relations among the values of its Fourier transform at different points. However, at the present time these questions are only in the initial stages of investigation.

The material of this fourth volume represents a complete unit in itself, and, as we have said, the exposition is practically independent of the preceding volumes. In spite of the relation of one chapter to another, one can begin a reading of this book with the first chapter, which contains the general theory of nuclear and rigged Hilbert spaces, or with the second chapter, which discusses the more elementary theory of positive definite generalized functions.

We mention that certain chapters contain, together with general results, others of a more specialized nature; these can be passed over at the first reading.

The authors wish to express their deep gratitude to those who helped them in working on this book: F. V. Shirokov, whose contributions far exceeded the limits of ordinary editorial work, A. S. Dynin, B. S. Mityagin, and V. B. Lidskii, whose valuable advice the authors used in writing up various topics in the first chapter. They express their particular thanks to S. A. Vilenkin, who took upon herself all the work connected with preparing the manuscript for press.

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N. Ya. Vilenkin

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CHAPTER I

THE KERNEL THEOREM. NUCLEAR SPACES. RIGGED HILBERT SPACE

This chapter is devoted to the study of a class of countably normed spaces[1]—so-called *nuclear spaces*. These spaces first appeared in connection with the "kernel theorem," which will be used repeatedly in this book. Later it became evident that nuclear spaces also play an essential role in many other topics of functional analysis, namely, nuclear spaces turn out to be the most natural class of spaces for the study of the spectral decomposition of self-adjoint operators. These spaces were already introduced in Volume III (Chapter IV, Section 3.1), in connection with the consideration of spectral decompositions. However, the definition of nuclearity which was given there is not entirely suitable for the study of other questions. Therefore in this volume we will use another definition of nuclear space, which in the essential cases is equivalent to nuclearity in the sense discussed in Volume III. The discussion in this chapter does not depend upon that in Volume III. In order to attain a complete independence of these treatments, we will present in this chapter certain results on the spectral decomposition of self-adjoint operators. Here, however, our main attention will be given to the general aspects of theory, unlike the treatment in Volume III, where a not inconsiderable part was played by the applications of these results to specific differential operators.

An important role is played by the concept of a rigged Hilbert space.

[1] We assume that the reader is familiar with the concept of a countably normed space to the extent of Chapters I and II of Volume II of this series. Besides, a brief discussion of the basic facts relating to a special case of such spaces—countably Hilbert spaces—is given at the beginning of Section 3. Let us note that throughout this volume every countably normed space considered will be taken, without special mention, to be complete.

Moreover, we will as a rule assume that the compatible norms $\| \varphi \|_n$, $1 \leqslant n < \infty$, which define the topology in a countably normed space Φ are monotonically increasing, i.e., that for every element $\varphi \in \Phi$ the inequalities

$$\| \varphi \|_1 \leqslant \cdots \leqslant \| \varphi \|_n < \cdots$$

hold.

This concept arises in considering nuclear spaces in which an inner product is introduced in some way or another. The theory of rigged Hilbert spaces is discussed in Section 4, where applications of this theory to the spectral analysis of self-adjoint operators are presented.

Also related to the theory of nuclear spaces is the subject of measure theory in linear topological spaces, discussed in Chapter IV. We will show in that chapter that the nuclearity of a space Φ is a necessary and sufficient condition for every measure on the cylinder sets in the space Φ', conjugate to Φ, to be completely additive.

1. Bilinear Functionals on Countably Normed Spaces. The Kernel Theorem

In this section we will study the general form of bilinear functionals[2] on countably normed spaces. We will show that any bilinear functional $B(\varphi, \psi)$, continuous with respect to its arguments φ and ψ which range over countably normed spaces Φ and Ψ, is continuous with respect to certain norms $\| \varphi \|_m$ and $| \psi |_n$ in these spaces. In other words, we will show that

$$| B(\varphi, \psi) | \leqslant M \| \varphi \|_m \, | \psi |_n$$

for all elements φ and ψ of the spaces Φ and Ψ, where the numbers M, m, n do not depend upon φ and ψ.

Applying this result to various specific spaces, we obtain the general form of bilinear functionals on these spaces. One of the most important results thereby obtained is a description of the bilinear functionals on the space K of infinitely differentiable functions with bounded supports. It will be proved that these functionals have the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)),$$

where F is a linear functional on the space K_2 of infinitely differentiable functions of the variables $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$, with

[2] A bilinear functional is a functional which is linear in both arguments φ and ψ. Further on we will encounter functionals $B(\varphi, \psi)$ which are linear in φ and antilinear in ψ, i.e.,

$$B(\alpha\varphi_1 + \beta\varphi_2, \psi) = \alpha B(\varphi_1, \psi) + \beta B(\varphi_2, \psi)$$

but

$$B(\varphi, \alpha\psi_1 + \beta\psi_2) = \bar{\alpha} B(\varphi, \psi_1) + \bar{\beta} B(\varphi, \psi_2) .$$

Such functionals are called Hermitean-bilinear, or simply Hermitean functionals.