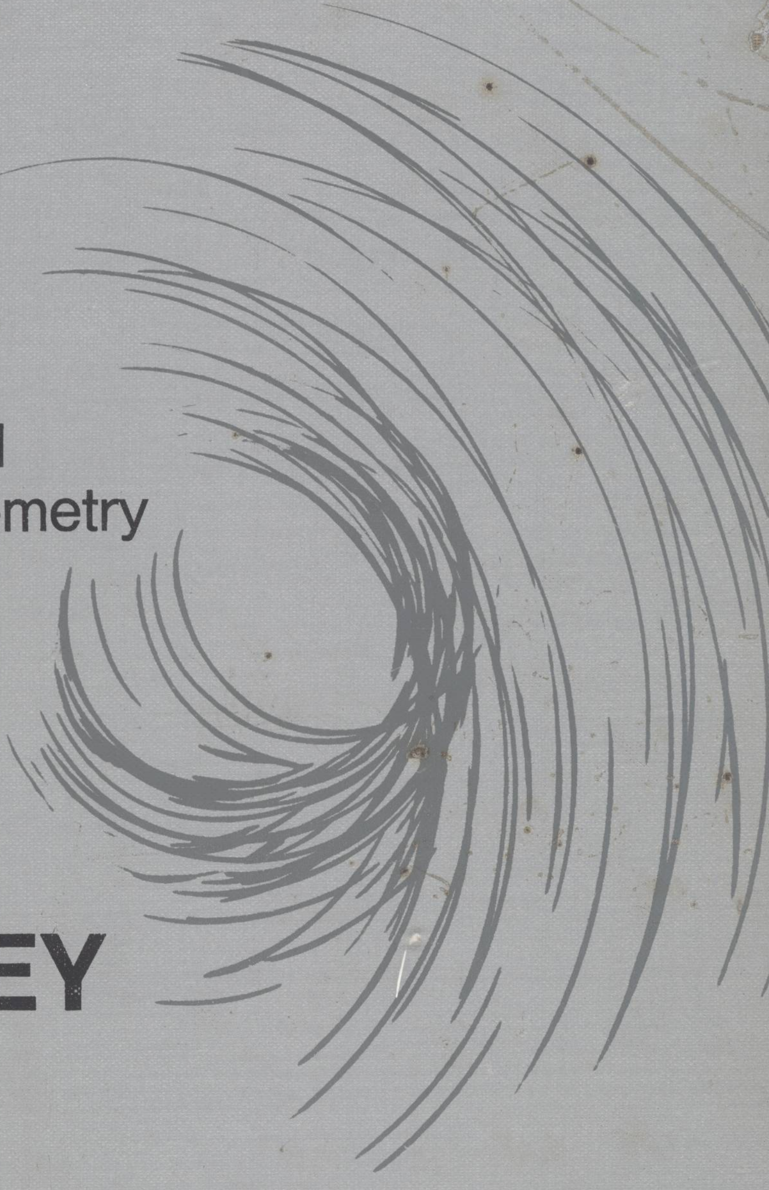
Second Edition

Calculus and Analytic Geometry



TIERNEY

Second Edition

CALCULUS
AND
ANALYTIC GEOMETRY

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Calculus and Analytic Geometry

TO MY WIFE

Preface

The second edition of CALCULUS AND ANALYTIC GEOMETRY employs the same approach as the first edition. It is addressed to the average student at a level of mathematical sophistication that will enable him to understand the basic concepts of analytic geometry and calculus, to see their relevance in science and technology, and to appreciate their beauty and elegance as intellectual creations. Prerequisites are high school algebra, geometry, and trigonometry.

Set notation and terminology are used in a manner consistent with our belief that beginning students are often confused by an excessively concise notation. The language is modern; we employ single letters to denote functions, we do not speak of a function being continuous or discontinuous at a point not in its domain, we do not use the terms "dependent" and "independent" variables, etc. Although we prefer to be up-to-date, we do not dwell upon these matters, since we recognize that usage varies in the mathematical community.

Although ϵ-δ definitions are carefully stated and discussed, manipulations of formal epsilonics are kept to a minimum. Many important theorems are stated and illustrated but not proved; references to proofs are given. This is consistent with the recommendations of the CUPM (Committee on the Undergraduate Program in Mathematics). The 1965 Committee Report to the Mathematical Association of America states that "it is the level of rigor in the student's understanding which counts and not only the rigor of the text or lecture presented to him."

In this new edition, the first chapter has been rewritten and now contains a short section on sets, a set of axioms for the real number system, a section on the algebra of functions, and an expanded treatment of inequalities. In Chap. 6 a treatment of rotation of axes has been included and in Chap. 8 a short section on work has been added. Other sections have been rewritten and some changes in the order of topics have been incorporated. The material can be covered in three semesters or in a shorter course if some topics are omitted. We have resisted the temptation to include a chapter on linear algebra; we feel that a minimum of two semester hours is required to do justice to this important subject. As the CUPM report points out, linear algebra is not required in a first calculus course unless one wishes to give a somewhat sophisticated presentation of multivariate calculus.

Although we do not assume that the student has access to a digital computer, we include many problems which can be solved by machine computation and we stress the importance of numerical analysis whenever possible.

The problem lists, which have been expanded, constitute an outstanding feature of the book. The importance of success in the learning process is recognized by starting these lists with many simple, routine problems. These are followed by problems of moderate difficulty involving ramifications of the basic theory. The lists conclude with problems which extend the theory and require considerable ingenuity. Numerous applications are included in all three of these categories. The selection of problems is based on the philosophy that the student is more apt to learn when he is successful, when his interest is aroused, when he sees the relevance of the subject matter, and when he is challenged.

Sequences are introduced in Chap. 1 and employed throughout the text. Chapter 2 develops the analytic geometry of straight lines and very simple curves. Differential and integral calculus are developed in Chaps. 3 and 4 and in later chapters the powerful methods of the calculus are applied to more advanced topics of analytic geometry. Thus the student is not confronted simultaneously with new concepts of both disciplines, and yet the two subjects are truly integrated.

Other features include thorough treatments of the function concept, inverse functions, the two fundamental theorems of calculus, and vector methods. Logical reasoning is stressed and yet strong reliance is placed upon geometrical intuition and interpretation. Many illustrative examples, arranged in order of complexity, are included; applications to physics, mechanics, chemistry, biology, economics, statistics, engineering, etc. are exploited; and considerable attention is devoted to the historical development of analytic geometry and calculus.

The author wishes to thank Professor J. H. Carruth of the University of Tennessee and Professor Erik Hemmingsen of Syracuse University for their many valuable suggestions.

It is my hope that the readers of this book will find their study of man's outstanding intellectual achievement a richly rewarding educational experience.

Annapolis, Maryland JOHN A. TIERNEY

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Preliminary Concepts

1.1 Introduction

This text presents an integrated treatment of *analytic geometry, differential calculus,* and *integral calculus.* Although it is difficult to discuss these three subjects before we have made certain intuitive concepts precise, it is nevertheless appropriate that we indicate the scope and importance of what lies ahead.

The basic problem of the integral calculus involves the calculation of a quantity Q which can be approximated by the sum of n terms $Q_1 + Q_2 + \cdots + Q_n$. If, when the number of addends increases indefinitely in such a manner that successive addends approach zero, the sum of the n addends approaches Q in value, we say that Q is the limit of the sum $Q_1 + Q_2 + \cdots + Q_n$. For example, the circumference of a circle is defined as the limit, as n increases indefinitely, of the sum of the lengths of the sides of an inscribed regular polygon of n sides. There are many other quantities which can be expressed in this manner as the limit of a sum; Q might be an area, a volume, a mass, a moment of inertia, the work done by a force, and so forth.

The basic problem of the integral calculus can be interpreted geometrically. In Fig. 1.1, AC and BD are $\perp CD$, and AB is an arc of a curve extending from A to B. The basic problem, that of computing the shaded area Q, is known as the *problem of quadrature.* This problem was solved in a number of special cases by Archimedes (287?–212 B.C.), the great mathematician of ancient times.

The French mathematician and philosopher René Descartes (1596–1650) is generally regarded as the founder of analytic geometry. His *La Géometrie,* published in 1637, interpreted algebraic operations

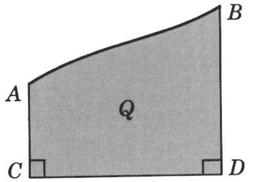

Figure 1.1

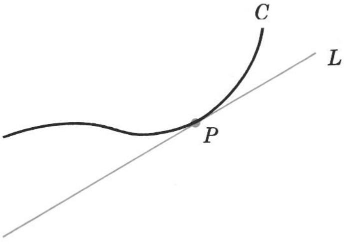

Figure 1.2

geometrically and applied algebraic methods to geometric problems. This approach furnished a highly powerful method for attacking geometric problems, and, due to the geometric interpretation of algebraic equations, led to the discovery of many new and interesting curves. (In Euclidean plane geometry, plane figures are bounded by straight line segments and arcs of circles.) Analytic geometry employed a coordinate system and provided the ideal setting for the development of the differential calculus.

In algebra we are often interested in the relation between two quantities, for example the temperature τ of a body and the time t. In the differential calculus we are concerned also with the rate at which τ is changing with respect to t. In fact, the basic problem of the differential calculus is to determine the instantaneous rate of change of one quantity with respect to another. Many important physical laws involve not only quantities but also their rates of change.

The basic problem of the differential calculus also has a geometric interpretation. In Fig. 1.2, P is a point on curve C. The basic problem is to determine the line L tangent to C at P. Clearly, this involves a generalization of the concept of a line tangent to a circle.

The *tangent problem* and the *quadrature problem* are seemingly unrelated; the first person to perceive their intimate connection was Isaac Barrow (1630–1677), the teacher of Sir Isaac Newton (1642–1727). Barrow recognized that the tangent and quadrature problems are *inverse problems*, in the same sense that addition and subtraction are inverse operations.

Newton and Gottfried Wilhelm Leibniz (1646–1716) are regarded as the founders of calculus. They appreciated and exploited the power and generality of Barrow's discovery and solved the quadrature problem by inverting the tangent problem. They also systematized the calculus into an organized body of mathematical knowledge. Of the many other men involved in the early development of our subjects, special mention should be made of Pierre de Fermat (1601–1665), a brilliant French mathematician who made many significant contributions to analytic geometry and to both branches of the calculus.

Although the origins of the calculus lie in the physical sciences, its modern applications permeate all of science and technology. Calculus is also the basis of the branch of mathematics known as *analysis* and is considered by many to be the outstanding intellectual achievement of the human race.

1.2 Sets

A *set* is a collection of well-defined, distinguishable objects regarded as a single entity. It is customary to denote a set by a capital letter and the objects in the set, termed *elements of the set,* by lowercase letters. A set is usually denoted by braces; for example, if set S contains elements a, b, c, and d, we write

$$S = \{a, b, c, d\}$$

or

$$S = \{b, c, a, d\}$$

since the order in which the elements are listed is unimportant.

A set may contain an infinite number of elements. For example, the set N of *positive integers* or *natural numbers* is denoted by

$$N = \{1, 2, 3, \ldots\}$$

where the three dots are read as "and so forth."

Another method of denoting a set S, known as the *set-builder notation,* is to write

$$S = \{x: p(x)\}$$

read as "S is the set of x's such that $p(x)$ is true," $p(x)$ referring to some property which holds for x. For example,

$$S = \{x: x \text{ is a positive integer less than } 6\}$$

denotes the set $S = \{1, 2, 3, 4, 5\}$.

If t is an element of set A, we write $t \in A$, but if t is not an element of A, we write $t \notin A$.

Equality of sets is defined as follows:

Definition: Two sets are equal if and only if they contain the same elements.

If set A is equal to set B, we write $A = B$ (or $B = A$), otherwise we write $A \neq B$. If every element of A is also an element of B, we say that A is contained in B, and we write $A \subseteq B$ and call A a *subset* of B. If $A \subseteq B$ but $A \neq B$, there exists at least one element t such that $t \in A$ and $t \notin B$. In this case we call A a *proper subset* of B and we write $A \subset B$.

It is convenient to postulate the existence of a set that contains no elements. This set is called the *empty set* and is denoted by $\varnothing$.

If all sets under consideration are subsets of a set $\mathbf{U}$, then $\mathbf{U}$ is called the *universal set* for that discussion. If $\mathbf{U} = N$ is the set of positive integers and A is the set of positive integers less than 100, we can write

$$A = \{x: x \in N \quad \text{and} \quad x < 100\}$$

However, it is simpler to write

$$A = \{x: x < 100\}$$

provided it is clear that A is a subset of N.

Example 1. Let $A = \{b, c, e, g\}$ and $B = \{a, b, c, e, f, g\}$. Then,

$$A \neq B, A \subset B, \quad b \in A, b \in B, \quad \text{and} \quad f \notin A$$

Sets can be combined to form new sets in the same fashion that numbers can be combined to yield other numbers. We define two operations on sets:

DEFINITION: The union of two sets A and B, written $A \cup B$ and read as "A union B," is the set of all elements which are members of A or B. In symbols

$$A \cup B = \{x: x \in A \quad \text{or} \quad x \in B\}$$

In this definition "$x \in A$ or $x \in B$" means that x is a member of at least one of A and B.

Example 2. Let $A = \{1, 3, 2\}$, $B = \{6, 3, 4, 2\}$, and $C = \{7, 8, 9\}$. Then,

$$A \cup B = \{1, 2, 3, 4, 6\} \quad \text{and} \quad A \cup C = \{1, 2, 3, 7, 8, 9\}$$

DEFINITION: The intersection of two sets A and B, written $A \cap B$ and read as "A intersect B," is the set of all elements which are members of both A and B. In symbols

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

Sets A and B are called *disjoint* if and only if $A \cap B = \varnothing$. That is, two sets are disjoint if and only if they have no common elements.

Example 3. Let $A = \{f, d, c, k, r\}$, $B = \{r, d, c, p\}$, and $C = \{p, q\}$. Then,

$$A \cap B = \{d, c, r\}, \quad B \cap C = \{p\}, \quad \text{and} \quad A \cap C = \varnothing$$

The totality of formulas and procedures for combining sets is known as *the algebra of sets*. It is similar to but not identical with the familiar algebra of the real or complex numbers. We now prove one formula, known as a *distributive law*, from the algebra of sets.

Example 4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution: Let $M = A \cup (B \cap C)$ and $N = (A \cup B) \cap (A \cup C)$.

(I) Let $x \in M$. Then, $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in N$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. Thus, $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in N$.

(II) Let $y \in N$. Then $y \in A \cup B$ and $y \in A \cup C$. Thus $y \in A$, or $y \in B$ and $y \in C$. If $y \in A$, then $y \in M$. If $y \in B$ and $y \in C$, then $y \in B \cap C$ and hence $y \in M$.

In (I) we showed that every element of M is also an element of N; in (II) we showed that every element of N is also an element of M. We conclude that M and N contain the same elements; that is, $M = N$.

The *theory of sets*, founded by the German mathematician Georg Cantor (1845–1918), is an important branch of mathematics and contains many deep and intriguing results. For our purposes, set notation and terminology will be useful in presenting the concepts of analytic geometry and calculus. Most sets we plan to consider will be subsets of the set $\mathscr{R}$ of *real numbers.*

Problem List 1.1

1. Let $A = \{a, b, c, d, e\}$, $B = \{b, c, d\}$, and $C = \{a, b, c, f\}$. Which of the following statements are true?
   (a) $b \in A$
   (b) $d \notin B$
   (c) $a \notin A \cap B$
   (d) $f \in B \cup C$
   (e) $d \in A \cap C$
   (f) $A \subset B$
   (g) $B \subset A$
   (h) $B \subseteq A$
   (i) $B \cap C = \varnothing$
   (j) $A \cup B \subset A$
   (k) $A \cup B \subseteq A$
   (l) $B \cap A \subset B$
2. List specifically the elements of each of the following sets:
   (a) $\{x: x$ is a New England State$\}$.
   (b) $\{x: x$ is an integer and $x^2 - x - 6 = 0\}$.
   (c) $\{x: x$ is a vowel in the English alphabet$\}$.
3. Let $A = \{1, 2, 3, 5, 8, 9\}$. List specifically the elements of the following subsets of A:
   (a) $\{x: x$ is odd$\}$
   (b) $\{x: x$ is even$\}$
   (c) $\{x: x$ is greater than 4$\}$
4. Employ the set-builder notation to denote the following sets. Let N denote the set of positive integers.
   (a) The set of positive integers less than 100
   (b) The set of odd positive integers
   (c) The set of positive integers which are squares of positive integers
5. Given $A = \{a, b, d, f, g\}$, $B = \{a, c, d, f\}$, and $C = \{b, c, d\}$, find
   (a) $A \cup B$ (b) $B \cup C$ (c) $A \cap B$ (d) $A \cap C$
6. Given $A = \{a, b, c, d, e\}$, $B = \{b, c, e, f\}$, and $C = \{a, b, d, f, g\}$, verify that
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) \neq (A \cup B) \cap C$
7. If A contains 9 elements, B contains 6 elements, and $A \cap B$ contains 2 elements, how many elements are there in $A \cup B$?
8. Prove that $\varnothing \subseteq S$ for every set S.
9. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
10. Let A and B be subsets of a universal set $\mathbf{U}$. Prove $\varnothing \subseteq A \cap B \subseteq A \cup B \subseteq \mathbf{U}$.
11. Give an example of a set whose elements are sets.
12. In what way can the perimeter of a triangle be considered a set whose elements are sets?
13. Explain the difference between t and $\{t\}$.
14. Explain why $\{a, b\} \notin \{a, b, \{a, c\}, \{a, b, c\}\}$.
15. Let A, B, and C be subsets of a universal set $\mathbf{U}$. Prove that
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
   (c) $A \cup (A \cap B) = A$
   (d) $A \cup \varnothing = A$
   (e) $A \cap \varnothing = \varnothing$
   (f) $A \cap A = A$
   (g) $A \cup (B \cup C) = (A \cup B) \cup C$
   (h) $A \cap (B \cap C) = (A \cap B) \cap C$