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André Weil

Elliptic Functions According to Eisenstein and Kronecker

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Editorial Preface

It is a great pleasure to my colleagues on the editorial board of the series Ergebnisse der Mathematik and myself to welcome this work, *Elliptic Functions according to Eisenstein and Kronecker*, by André Weil. However, some readers may be surprised to find in this series a text which appears at first sight to be in essence a contribution to the history of mathematics, and which would therefore seem to be very untypical of the books in this series. However, the editors are strongly of the opinion that, while this text undoubtedly contributes notably to the history of our science, it is also of great value to contemporary mathematical research. Thus we had no hesitation in asking the permission of Professor Weil to publish his text in our series; and we are delighted to have had his agreement to do so.

PETER HILTON
Chairman, Editorial Board, Ergebnisse der Mathematik

Battelle Seattle Research Center, August, 1975

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Part I

EISENSTEIN

Chapter I

Introduction

In 1891, Kronecker agreed to give the inaugural lecture at the first meeting of the newly founded German Mathematical Society. He cancelled this plan after losing his wife, but (in a letter to Cantor, president of the society) expressed the hope that he would still be able to supply a written text for the lecture, which he described as follows:

„Der Vortrag ... sollte kurzweg den Titel haben „Über Eisenstein" ... Dabei müßten dann außer den rein arithmetischen und analytisch-arithmetischen noch ganz besonders seine rein analytischen Untersuchungen über elliptische Funktionen hervorgehoben werden, welche dem Bewußtsein der Jetztzeit ganz abhanden gekommen sind ..."[1]

Soon after that, Kronecker died; he never wrote up that lecture. However, he had already discussed Eisenstein's work at some length (pointing out how Eisenstein had anticipated some of Weierstrass' best-known innovations and gone well beyond them) in his last major paper on elliptic functions, printed in 1891 by the Berlin Academy. This is how he comments upon it:

"Essentially new points of view ... particularly concerning the transformation theory of theta-functions ... were introduced by Eisenstein in the fundamental but seldom quoted *Beiträge zur Theorie der elliptischen Functionen* published in *Crelles Journal* in 1847, which are based upon entirely original ideas ..."

When Kronecker expresses himself with such enthusiasm, it is rather obvious that he has only just re-discovered the paper in question; he goes on to point out its relevance to his current research, which he had clearly not noticed until that moment[2]. In both of the above quotations, the reference is to Eisenstein's paper *Genaue Untersuchung der unendlichen Doppelproducte, aus welchen die elliptischen Functionen als Quotienten zusammengesetzt sind, und der mit*

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[1] "The lecture was to be entitled simply "On Eisenstein". Its theme was to be, not so much his work on number-theory or those of his papers which combine number-theory with function-theory, but more specifically, and with particular emphasis, his purely analytical investigations on elliptic functions, which have been so profoundly forgotten that they now seem to be as good as lost." (Kronecker, *Werke*, vol. V, p. 499.)

[2] Kronecker, *Werke*, vol. V, p. 149.

ihnen zusammenhangenden Doppelreihen; this is part VI of his *Beiträge zur Theorie der elliptischen Functionen*; it was published in vol. 35 (1847) of *Crelles Journal*, pp. 153—274, and reproduced in Eisenstein's volume *Mathematische Abhandlungen* published in 1847 with a preface by Gauss.

Well could Kronecker say of that paper that it was "seldom quoted"; it is doubtful whether there is a single reference to it, apart from Kronecker's and from a footnote in Hurwitz' thesis[3], in all the mathematical literature of the XIXth century; in the present century one could perhaps find two or three more. Eisenstein's ideas could indeed seem "as good as lost".

It is not merely out of an antiquarian interest that the attempt will be made here to resurrect them. Not only do they provide the best introduction to much of the work of Hecke; but we hope to show that they can be applied quite profitably to some current problems, particularly if they are used in conjunction with Kronecker's late work which is their natural continuation. Perhaps their range of usefulness can even be extended beyond the theory of elliptic functions and of the modular group, and in particular to the arithmetical study of the Eisenstein series for the Hilbert modular group[4]; but this will not be discussed here.

As any reader of Eisenstein must realize, he felt hard pressed for time during the whole of his short mathematical career. As a young man he complains of nervous ailments which often compel him to interrupt his work; later, he developed tuberculosis, and died of it in 1852 at the age of 29. His papers, although brilliantly conceived, must have been written by fits and starts, with the details worked out only as the occasion arose; sometimes a development is cut short, only to be taken up again at a later stage. Occasionally Crelle let him send part of a paper to the press before the whole was finished. One is frequently reminded of Galois' tragic remark "Je n'ai pas le temps".

In view of this, it would be foolish to follow Eisenstein step by step; we shall feel free to re-arrange his material as he might have done himself on more mature consideration, and to make use of his own hints in order to improve upon his exposition when this does no violence to his way of thinking.

One general observation is needed concerning questions of convergence. In Eisenstein's days, the concept of absolute convergence (as opposed to "conditional" convergence) was still comparatively new; he learnt it, he says, from Dirichlet's paper on the arithmetic progression, and he is quite careful in using it whenever needed. For instance, the earlier part of the paper to be studied here is devoted to a proof for the convergence of the series

$$\sum(n_1^2+n_2^2+\cdots+n_\nu^2)^{-\sigma}$$

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[3] Hurwitz, *Werke*, vol. I, p. 31.

[4] This prediction has been fulfilled (even beyond my own expectations) since the above lines were written; cf. G. Shimura, *On some arithmetic properties of modular forms of one and several variables*, to appear in *Ann. of Math.*

with $\sigma > \nu/2$, and of more general series of the same kind. Nowadays such results are a matter of common knowledge and may be taken for granted. On the other hand, Eisenstein is unaware of the concept of uniform convergence; he assumes tacitly, and without proof, that the series of analytic functions which he introduces can be differentiated term by term; perhaps this was why Weierstrass ignored his work. Actually the gap is easily filled; if challenged to do so, Eisenstein might well have proceeded as follows. Take as a typical example the case of the series $\sum(x+\mu)^{-n}$ which occur in his theory of the trigonometric functions (cf. Chap. II). Discarding a finite number of terms, we have to consider the absolutely convergent series

$$f_n(x) = \sum_{\mu=M}^{+\infty}(x+\mu)^{-n} + \sum_{\mu=M}^{+\infty}(x-\mu)^{-n} \qquad (n \geqq 2)$$

$$f_1(x) = \sum_{\mu=M}^{+\infty}\left(\frac{1}{x+\mu} + \frac{1}{x-\mu}\right),$$

where M is an integer >1. Call $f(x)$ any one of these series; write $\varphi_\mu(x)$ for its μ-th term, and expand $\varphi_\mu(x+y)$, by the binomial formula, into a power-series in y:

$$\varphi_\mu(x+y) = \sum_{m=0}^{+\infty}\varphi_{\mu,m}(x)\,y^m .$$

A trivial estimate will then show that the double series $\sum_{\mu,m}\varphi_{\mu,m}(x)\,y^m$ is absolutely convergent for $|x| \leqq M-1$, $|y|<1$. Consequently we can write

$$f(x+y) = \sum_{m=0}^{+\infty}\left(\sum_{\mu}\varphi_{\mu,m}(x)\right)y^m .$$

As the coefficients of the y^m are nothing else, up to obvious constant factors, than the series derived from the series for $f(x)$ by successive differentiation term by term, this guarantees the legitimacy of Eisenstein's procedure. In what follows, such matters will be taken for granted once and for all.

Chapter II

Trigonometric Functions

§ 1. As Eisenstein shows, his method for constructing elliptic functions applies beautifully to the simpler case of the trigonometric functions. Moreover, this case provides, not merely an illuminating introduction to his theory, but also the simplest proofs for a series of results, originally discovered by Euler, which will have to be used later on.

The method is based on the consideration of the series

$$\varepsilon_n(x) = \sum_{\mu=-\infty}^{+\infty} (x+\mu)^{-n},$$

where n is an integer $\geqq 1$. This requires no comment if $n>1$. In order to deal with the case $n=1$, we introduce a symbol $\sum_e$, to be called "Eisenstein summation" (for simple series), defined by

$$\sum_{\mu}{}_e = \lim_{M \to +\infty} \sum_{\mu=-M}^{+M}.$$

Then we define ε_1 by putting

$$\varepsilon_1(x) = \sum_{\mu}{}_e \frac{1}{x+\mu}.$$

As the series for ε_n is absolutely convergent for $n \geqq 2$, it is obvious that, for such n, ε_n is periodic of period 1; the same is true of ε_1 because the terms of the series for ε_1 tend to 0 for $\mu \to \pm\infty$. Differentiating term by term (cf. the final remarks in Chap. I), we get $d\varepsilon_n/dx = -n\varepsilon_{n+1}$ for all $n \geqq 1$. Expanding $(x+\mu)^{-n}$, for $n \geqq 1$, $\mu \neq 0$, $|x|<1$, into a power-series in x, we find for $\varepsilon_n(x) - x^{-n}$ a power-series in x, convergent for $|x|<1$. For $n=1$, we will write this as

$$\varepsilon_1(x) = \frac{1}{x} - \sum_{m=1}^{+\infty} \gamma_m x^{m-1}$$

where the coefficients γ_m are 0 when m is odd, and otherwise are given by

$$\gamma_{2m} = 2 \sum_{\mu=1}^{+\infty} \mu^{-2m}.$$

Differentiating $n-1$ times, we get:

$$\varepsilon_n(x) = \frac{1}{x^n} + (-1)^n \sum_{m=1}^{+\infty} \binom{2m-1}{n-1} \gamma_{2m} x^{2m-n},$$

where the "binomial coefficients" $\binom{2m-1}{n-1}$ are 0 for $2m<n$. Clearly $\varepsilon_n(x)$ is an even or an odd function of x according as n is even or odd. For each $n \geqq 1$, γ_n is the value of $\varepsilon_n(x) - x^{-n}$ at $x=0$.

§ 2. The question is now to construct non-linear identities between the functions ε_n; the starting point for this is supplied, according to Eisenstein, by identities for rational functions. Take two independent variables p, q, and put $r=p+q$. Dividing by pqr, we get

$$\frac{1}{pq} = \frac{1}{pr} + \frac{1}{qr}. \tag{1}$$

More generally, if m, n are two integers $\geqq 1$, we have

$$\frac{1}{p^m q^n} = \sum_{h=0}^{m-1} \frac{n(n+1)\dots(n+h-1)}{h!\, p^{m-h} r^{n+h}} + \sum_{k=0}^{n-1} \frac{m(m+1)\dots(m+k-1)}{k!\, q^{n-k} r^{m+k}}. \tag{2}$$

This can be derived from (1) by successive differentiation, $m-1$ times with respect to p and $n-1$ times with respect to q. Alternatively, (2) may be regarded as the partial fraction decomposition of $p^{-m}(r-p)^{-n}$ regarded as a rational function of p when r is taken as a constant. For $m=n=2$, we get:

$$\frac{1}{p^2 q^2} = \frac{1}{p^2 r^2} + \frac{1}{q^2 r^2} + \frac{2}{p r^3} + \frac{2}{q r^3}. \tag{3}$$

In (3), put $p=x+\mu$, $q=y+\nu-\mu$; also, put $z=x+y$; then $r=z+\nu$. Apply now "Eisenstein summation" to (3) with respect to μ, while ν is kept constant. This gives:

$$\sum_{\mu}{}_e (p^{-2}q^{-2} - p^{-2}r^{-2} - q^{-2}r^{-2}) = 2r^{-3}[\varepsilon_1(x) + \varepsilon_1(y+\nu)],$$

where we may replace $\varepsilon_1(y+\nu)$ by $\varepsilon_1(y)$, and $\sum_e$ by $\sum$ since the series is absolutely convergent. Now summation with respect to ν gives

$$\varepsilon_2(x)\varepsilon_2(y)-\varepsilon_2(x)\varepsilon_2(z)-\varepsilon_2(y)\varepsilon_2(z)=2\varepsilon_3(z)[\varepsilon_1(x)+\varepsilon_1(y)] \tag{4}$$

since everything now is absolutely convergent. This may be regarded as an addition formula for the ε-functions.

§ 3. For a given x, not an integer, both sides of (4), regarded as functions of y, have a double pole at $y=0$; expanding them into power-series in y, one verifies at once that the terms in y^{-2} and y^{-1} have the same coefficients on both sides. Equating the constant terms, we get

$$3\varepsilon_4(x)=\varepsilon_2(x)^2+2\varepsilon_1(x)\varepsilon_3(x). \tag{5}$$

Similarly, for a given x, regard both sides of (4) as functions of z and expand them at $z=0$; equating the constant terms, we get

$$\varepsilon_2(x)^2=\varepsilon_4(x)+2\gamma_2\varepsilon_2(x). \tag{6}$$

This gives $\varepsilon_1\varepsilon_3=\varepsilon_2^2-3\gamma_2\varepsilon_2$; differentiating this, we get $\varepsilon_2\varepsilon_3-2\gamma_2\varepsilon_3=\varepsilon_1\varepsilon_4$; combined with (6), this gives $\varepsilon_3=\varepsilon_1\varepsilon_2$. Substituting $\varepsilon_1\varepsilon_2$ for ε_3 in the formula for $\varepsilon_1\varepsilon_3$ and dividing by ε_2, we get $\varepsilon_1^2=\varepsilon_2-3\gamma_2$. As we have $\varepsilon_2=-d\varepsilon_1/dx$, this shows that ε_1 is the solution of the differential equation $dX/dx=-X^2-3\gamma_2$ which becomes infinite at $x=0$. It is well known, of course, that this implies $\varepsilon_1(x)=\pi\cot\pi x$ (we write cot for the cotangent).

§ 4. It is more interesting, however, to proceed as if one had no previous knowledge of the trigonometric functions, and to regard the differential equation in question as defining the cotangent. More precisely, put $a=(3\gamma_2)^{-\frac{1}{2}}$, $a>0$. From the above construction, it follows that $u\to a\varepsilon_1(au)$ is a solution of the differential equation $dv/du=-u^2-1$, with the period a^{-1}; it is clear that any two solutions of that equation can differ only by a translation on u; therefore this equation has a unique solution which becomes infinite for $u=0$. If we *define* this solution to be $v=\cot u$, and if we *define* π to be its period, we can now write $\varepsilon_1(x)=\pi\cot\pi x$, and we have $\gamma_2=\pi^2/3$.

From this point on, one can develop in various manners the elementary theory of the trigonometric functions; this can be done either by making use of the formulas obtained above or by deriving further identities for trigonometric functions from identities for rational functions according to the method described in § 2. As an example, take the addition formula for the cotangent:

$$2\varepsilon_1(x+y)[\varepsilon_1(x)+\varepsilon_1(y)]=[\varepsilon_1(x)+\varepsilon_1(y)]^2-\varepsilon_2(x)-\varepsilon_2(y). \tag{7}$$

Eisenstein proves this as follows. First observe that, for every integer ν, we have

(8) $$\varepsilon_1(x)+\varepsilon_1(y)=\sum_{\mu}\left(\frac{1}{x+\mu}+\frac{1}{y+\nu-\mu}\right),$$

the series being absolutely convergent. Put now

$$z=x+y,\quad p=x+\mu,\quad q=y-\mu,\quad p'=x+\mu-\nu,\quad q'=y+\nu-\mu,$$

and apply (1) to p,q' and to p',q. We get

$$\frac{1}{pq'}+\frac{1}{p'q}=\frac{1}{p+q'}\left(\frac{1}{p}+\frac{1}{q'}\right)+\frac{1}{p'+q}\left(\frac{1}{p'}+\frac{1}{q}\right),$$

which may be regarded as the partial fraction decomposition of the left-hand side when it is regarded as a function of x alone, z,μ and ν being kept constant. Write A for the right-hand side. Similarly, applying (1) to p, $-p'$ and then to q, $-q'$, we get, for $\nu\neq 0$:

$$\frac{1}{pp'}+\frac{1}{qq'}=\frac{1}{\nu}\left(\frac{1}{p'}-\frac{1}{p}+\frac{1}{q}-\frac{1}{q'}\right);$$

write B_ν for the right-hand side, and put $B_0=p^{-2}+q^{-2}$. This gives the identity

$$\left(\frac{1}{p}+\frac{1}{q}\right)\left(\frac{1}{p'}+\frac{1}{q'}\right)=A+B_\nu.$$

Sum this with respect to μ, ν being kept constant; then perform Eisenstein summation, i.e. $\sum_e$, on ν; in the first summation, everything is absolutely convergent. Moreover, in the left-hand side, even double summation, on (μ,ν), would be absolutely convergent; therefore, in the left-hand side, one would get the same result by performing summation on μ and on $\mu-\nu$ independently, which, in view of (8), gives

$$[\varepsilon_1(x)+\varepsilon_1(y)]^2.$$

As to the summation on A, it gives the left-hand side of (7); the summation on B_0 gives the last two terms in (7), and the summation on B_ν for $\nu\neq 0$ gives 0. This proves (7).

Alternatively, consider the formula

(9) $$\varepsilon_1(z-x)[\varepsilon_2(x)-\varepsilon_2(z)]-\varepsilon_1(x)\varepsilon_2(z)-\varepsilon_1(z)\varepsilon_2(x)=0,$$

which, in view of the identity $\varepsilon_2=\varepsilon_1^2+\pi^2$ of § 3, is trivially equivalent to (7). Put $y=z-x$, and write $f(x,y)$ for the left-hand side of (9); then (4) gives $\partial f/\partial y=0$, so that, instead of $f(x,y)$, we may write $f(x)$. As this left-hand side is symmetric in x and z, we have $f(x)=f(z)$ for all x, z, so that f is constant; changing x, z into $-x, -z$, the left-hand side changes sign, so that f is odd. Therefore $f=0$; this again proves (7).

§ 5. We merely note here that § 3 of Eisenstein's paper contains brief indications about far more general identities for trigonometric functions which can be deduced by his method from the corresponding identities for rational functions and in their turn (as we shall see) can be used in order to derive identities for elliptic functions. We also note his incidental mention of the formula

$$\frac{1}{\sin u}=\frac{1}{2}\cot\frac{u}{2}-\frac{1}{2}\cot\frac{u+\pi}{2},$$

which can be rewritten as

$$\frac{\pi}{\sin\pi x}=\frac{1}{2}\varepsilon_1\left(\frac{x}{2}\right)-\frac{1}{2}\varepsilon_1\left(\frac{1+x}{2}\right)=\sum\nolimits_e\frac{(-1)^\nu}{x+\nu}. \tag{10}$$

This may be regarded as *defining* the sine function. Now (7) gives the multiplication formula

$$2\varepsilon_1(2x)\varepsilon_1(x)=2\varepsilon_1(x)^2-\varepsilon_2(x)=\varepsilon_1(x)^2-\pi^2.$$

Here substitute either $\frac{x}{2}$ or $\frac{1+x}{2}$ for x; this shows that $\varepsilon_1\left(\frac{x}{2}\right)$ and $\varepsilon_1\left(\frac{1+x}{2}\right)$ are the roots of the equation

$$Y^2-2\varepsilon_1(x)\,Y-\pi^2=0,$$

and (10) gives

$$\pi/\sin\pi x=\varepsilon_2(x)^{\frac{1}{2}},\qquad \varepsilon_2(x)=(\pi/\sin\pi x)^2.$$

Differentiating this, and using the formulas of § 3, one gets

$$\varepsilon_1(x)=\frac{d}{dx}\log\sin\pi x. \tag{11}$$

Had Eisenstein lived longer and pursued his investigations further, he might well have been led by (10) and other formulas of the same kind to the consideration of the more general series of the form $\sum\chi(\nu)(x+\nu)^{-n}$, where χ is a character (not necessarily of finite order) of the additive group of integers, and of the corresponding series in the theory of elliptic functions. As we shall see, this topic was eventually taken up by Kronecker (cf. Chap. VII and VIII).

§ 6. Now we introduce infinite products; this calls for some preliminary remarks. Let $P = \prod p_\mu$ be such a product, where some of the p_μ may be 0; we extend to it all our definitions and remarks concerning series by taking logarithms, with the understanding that finitely many factors should always be disregarded when this is appropriate. This implies that the product is not called convergent unless p_μ tends to 1 for $\mu \to \pm\infty$ (so that only finitely many factors can be 0); then, for $\log p_\mu$, one always understands the principal branch of the log in some neighborhood of 1; outside that neighborhood, one may take for $\log p_\mu$ any branch of the log (e.g., for definiteness, the one which is of the form $\log|p_\mu| + \pi i t$ with $-1 \leqq t < 1$), with the understanding that $\log 0 = \infty$; that being so, we write

$$\log P = \sum \log p_\mu$$

when the product converges; the log in the left-hand side may then be any branch of the log function. The product is said to be absolutely convergent if the series $\sum \log p_\mu$ is so (here one has to disregard the finitely many factors $p_\mu = 0$). The symbol $\prod_e$ has to be understood accordingly; in view of our definition of $\sum_e$, it may be regarded as defined by the formula

$$\prod_\mu{}_e\, p_\mu = \lim_{M \to +\infty} \prod_{\mu=-M}^{+M} p_\mu = \prod_{\mu=-M}^{+M} p_\mu \cdot \prod_{\mu=M+1}^{+\infty} (p_\mu p_{-\mu}),$$

this being meaningful only if $p_\mu p_{-\mu}$ tends to 1 for $\mu \to +\infty$ (while p_μ need not do so).

It is now meaningful to consider the product

$$P(x) = \prod_\mu{}_e' \left(1 + \frac{x}{\mu}\right),$$

where $\prod'$, as usual, means the product taken over all $\mu \neq 0$; moreover, we may write

$$\log P(x) = \sum_\mu{}_e' \log\left(1 + \frac{x}{\mu}\right).$$

In view of the remarks at the end of Chap. I, we may differentiate term by term; at this point, however, Eisenstein feels the need for some justification and proceeds, as one might do nowadays, by differentiating formally and then integrating the formula obtained in this manner; he fails to notice that even this falls short