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The Geometric and Arithmetic Volume of Shimura Varieties of Orthogonal Type

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The Geometric and Arithmetic Volume of Shimura Varieties of Orthogonal Type

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CHAPTER 1

Overview

1.1. Introduction

A Shimura variety of orthogonal type arises from the Shimura datum consisting of the orthogonal group $\mathrm{SO}(L_{\mathbb{Q}})$ of a quadratic space $L_{\mathbb{Q}}$ of signature $(l-2,2)$ and the set

$$\mathbb{D} := \{N \subseteq L_{\mathbb{R}} \mid N \text{ oriented negative definite plane}\}$$

which has the structure of an Hermitian symmetric domain and can be interpreted as a conjugacy class of morphisms $\mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_{\mathrm{m}} \to \mathrm{SO}(L_{\mathbb{R}})$. For any compact open subgroup $K \subseteq \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ we can form the Shimura variety

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D} \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K]$$

in which we always consider the quotient as an orbifold. It is a smooth manifold for sufficiently small K, and it always has an algebraic model M^K, in general a smooth Deligne–Mumford stack.[1]

For simplicity we assume for the moment that there is an integral lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ with cyclic discriminant group $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ of square-free order ε. Let K be the group of those integral isometries that induce the identity of the discriminant group. It goes back to Siegel [60] that in this case

$$\text{(1.1)} \quad \mathrm{vol}(\mathrm{M}^K) = 2^{-\lfloor (l-4)/2 \rfloor} \left(\prod_{i=1}^{\lfloor (l-1)/2 \rfloor} \zeta(1-2i) \right) \cdot \begin{cases} L\left(1-\frac{l}{2}, \chi\right) & 2 \mid l, \\ (-1)^{(l^2-1)/8} \prod_{p \mid \epsilon/2} \left(p^{(l-1)/2} + \left(\frac{-1}{p}\right)^{(l-1)/2} \Phi_p \right) & 2 \nmid l, \end{cases}$$

where χ is the associated quadratic character (possibly trivial), Φ_p is the Hasse invariant at p, and where the volume is understood w.r.t. to a natural volume form (highest power of the Chern class of a canonical ample automorphic line bundle). This volume is in fact an intersection number and should therefore be a rational number. Inserting the well-known formulas

$$\zeta(1-2i) = -\frac{B_{2i}}{2i}, \quad L\left(1-\frac{l}{2}, \chi\right) = -\frac{B_{l/2,\chi}}{l/2},$$

we see that the quantity (1.1) is indeed a rational number. It is also (up to a factor 2) the proportionality factor that occurs in the famous proportionality principle of Hirzebruch and Mumford [55] (cf. 2.6.22) applied to this case. For $l = 2$,

[1] defined over $\mathbb{Q}$ for $l > 2$

equation (1.1) is nothing but the classical analytic class number formula for definite binary quadratic forms. There exist formulas of similar shape for all locally symmetric spaces (see, e.g., [47] for the case of Chevalley groups). In the special case considered here M^K is, in fact, canonically defined over[2] $\mathbb{Z}[\frac{1}{2}]$. Therefore, using Arakelov theory, one obtains an analogous arithmetic 'volume element' (the highest power of the *arithmetic* Chern class of the same automorphic line bundle). To compute its Arakelov degree, which naturally is called the *arithmetic volume* of M^K, is the main objective of this book. The result is (in the special case considered here):

$$\widehat{\mathrm{vol}}(\mathrm{M}^K) \equiv \mathrm{vol}(\mathrm{M}^K) \cdot \left(\frac{l-1}{2}C + \sum_{i=1}^{\lfloor (l-1)/2 \rfloor} \left(-2\frac{\zeta'(1-2i)}{\zeta(1-2i)} - N_{2i} \right) + \begin{cases} -\frac{L'(1-l/2,\chi)}{L(1-l/2,\chi)} - \frac{1}{2}N_{l/2} - \frac{1}{2}\log\left|\frac{\varepsilon}{2}\right| & 2 \mid l, \\ \frac{1}{2}\sum_{p \mid \epsilon/2} \frac{p^{(l-1)/2} - \left(\frac{-1}{p}\right)^{(l-1)/2}\Phi_p}{p^{(l-1)/2} + \left(\frac{-1}{p}\right)^{(l-1)/2}\Phi_p} \log(p) + \frac{1}{2}\log(2) & 2 \nmid l, \end{cases} \right) \tag{1.2}$$

modulo rational multiples of $\log(2)$ (see B.1.3 for missing definitions).

We observe, of course, an apparent similarity between the formulas (1.1) and (1.2), and in fact, we have:

$$\mathrm{vol}(\mathrm{M}^K) = 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};0) \quad \widehat{\mathrm{vol}}(\mathrm{M}^K) \equiv 4\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\Big|_{s=0}$$

for the function

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s) = 2^{-\lfloor l/2 \rfloor}(2\pi)^{s(l-1)/2}|\varepsilon|^{-s/2} \prod_{i=1}^{\lfloor (l-1)/2 \rfloor} \frac{\Gamma(s+2i)\zeta(1-2s-2i)}{\Gamma(2s+2i)\cos(\pi(s+i))} \cdot \begin{cases} 2^{s/2}\frac{\Gamma(s/2+l/2)L(1-s-l/2,\chi)}{\Gamma(s+l/2)\cos(\pi(s/2+\lfloor l/4 \rfloor))} & \text{if } l \text{ is even}, \\ 2^s \prod_{p \mid \epsilon/2} \left(p^{(l-1)/2+s} + \left(\frac{-1}{p}\right)^{(l-1)/2} \Phi_p \right) & \text{if } l \text{ is odd}, \end{cases}$$

The function $\lambda^{-1}(L_{\mathbb{Z}};s)$ has, however, an intrinsic and much more general definition in terms of representation densities. Its study was the main subject of the paper [26] by the author. Formula (1.2) had been known only in some cases for $l \le 4$, and was conjectured in some cases for $l = 5$.

Our proof uses only information from the "Archimedean fibre," that is, we do not need explicit computations of local intersection numbers. We generalize work of Bruinier, Burgos, and Kühn [10] which dealt with the case of Hilbert modular surfaces. Borcherds' construction of orthogonal modular forms [2], and a computation of the integral of their norm [11,35] are used.

[2] and conjecturally over $\mathbb{Z}$

The orthogonal Shimura varieties are interesting in particular because they contain special algebraic cycles of arbitrary codimension whose arithmetic, respectively geometric volumes are conjectured to be encoded as a special value, respectively derivative of Fourier coefficients of Eisenstein series. This relation, in turn, has deep arithmetic consequences, including for example the formula of Gross and Zagier [21] and vast generalizations of it. It is also the key to the calculation of geometric and arithmetic volumes.

More precisely, these cycles are constructed as follows: For an isometry $x\colon M \hookrightarrow L$, where M is positive definite consider the subset

$$\mathbb{D}_x := \{N \in \mathbb{D} \mid N \perp x(M)\}$$

of $\mathbb{D}$. If (K-stable) lattices $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$ or more generally a K-invariant Schwartz function $\varphi \in S(M^*_{\mathbb{A}^{(\infty)}} \otimes L_{\mathbb{A}^{(\infty)}})$ is chosen, we define cycles $\mathrm{Z}(L, M, \varphi; K)$ on the Shimura variety by taking the quotient of the union of the $\mathbb{D}_x$ over all integral isometries (respectively all isometries in the support of φ in a weighted way). See Section 3.1 for the precise definition. For singular lattices M analogous cycles can be defined. Consider a model $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ of a toroidal compactification of the Shimura variety (the notation will be explained below). We consider the generating series

$$\Theta_m(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q \rangle, \varphi; K)] \exp(2\pi i Q \tau)$$

with values in its algebraic Chow group $\mathrm{CH}^m\left(\mathrm{M}({}^K_\Delta\mathbf{O}(L))_{\mathbb{C}}\right) \otimes \mathbb{C}$. Assume now that $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ is even defined over $\mathbb{Z}$ in a "reasonably canonical" way. Kudla proposes a definition of arithmetic cycles $\widehat{\mathrm{Z}}(L, M, \varphi; K, \nu)$, depending on the imaginary part ν of τ, too, and also for singular and for indefinite M, such that

$$\widehat{\Theta}_m(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\widehat{\mathrm{Z}}(L, \langle Q \rangle, \varphi; K, \nu)] \exp(2\pi i Q \tau)$$

should have values in a suitable Arakelov Chow group $\widehat{\mathrm{CH}}^m\left(\mathrm{M}({}^K_\Delta\mathbf{O}(L))\right) \otimes \mathbb{C}$. He proposes specific Green's functions which have singularities at the boundary.

The orthogonal Shimura varieties come equipped with a natural Hermitian automorphic line bundle $\Xi^*\overline{\mathcal{E}}$ whose metric also has singularities along the boundary. This provides us (via multiplication with a suitable power of its first Chern class and taking pushforward) with geometric (respectively arithmetic) degree maps $\deg\colon \mathrm{CH}^p(\cdots) \to \mathbb{Z}$ (respectively $\widehat{\deg}\colon \widehat{\mathrm{CH}}^p(\cdots) \to \mathbb{R}$). Assuming that an Arakelov theory can be set up to deal with all different occurring singularities, Kudla conjectures (for the geometric part this goes back to Siegel, Hirzebruch, Zagier, Kudla–Millson, Borcherds, etc.)[3]:

(K1) Θ_m and $\widehat{\Theta}_m$ are (holomorphic, respectively nonholomorphic) Siegel modular forms of weight $l/2$ and genus m.

(K2) $\deg(\Theta_m)$ and $\widehat{\deg}(\widehat{\Theta}_m)$ are equal to a special value (respectively the special derivative at the same point) of a *normalization* of the standard Eisenstein series of weight $l/2$ associated with the Weil representation of L [26, Section 4].

[3] If $l - r \leq m + 1$, the statement has to be modified. Here r is the Witt-rank of L.

(K3) $\Theta_{m_1}(L,\varphi_1;\tau_1)\cdot\Theta_{m_2}(L,\varphi_2;\tau_2)=\Theta_{m_1+m_2}\big(L,\varphi_1\oplus\varphi_2,\left(\begin{smallmatrix}\tau_1 & \\ & \tau_2\end{smallmatrix}\right)\big)$ and similarly for $\widehat{\Theta}$.

(K4) $\widehat{\Theta}_{l-1}$ can be defined with coefficients being zero-cycles on the arithmetic model, and it satisfies the properties above.

Kudla shows (see [36] for an overview) that this implies almost formally vast generalizations of the formula of Gross–Zagier [21]. In full generality the conjectures are known only for Shimura curves [44]. Section 1.3 contains a brief overview on what is known in other special cases.

(K2) is closedly related to the calculation of geometric and arithmetic volumes because the cycles in question consist themselves (for sufficiently good reduction) of models of orthogonal Shimura varieties of smaller dimension.

We are thus able to obtain partial results towards the *arithmetic part* of (K2) for *all* Shimura varieties of orthogonal type. More precisely, we prove the following:

3.5.5. Main Theorem. *Let $L_{\mathbb{Z}}\subset L_{\mathbb{Q}}$ be an integral lattice in a quadratic space of signature $(l-2,2)$. Let K be its discriminant kernel. Let D' be the product of the primes p such that $p^2\mid D$, where D is the discriminant of $L_{\mathbb{Z}}$. We have*

(1) $\mathrm{vol}_{\mathcal{E}}\Big(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}})\big)\Big)=4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};0)$,

(2) $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}\Big(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}})\big)\Big)\equiv\frac{\mathrm{d}}{\mathrm{d}s}4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s)|_{s=0}$ *in* $\mathbb{R}_{2D'}$.

Let $M_{\mathbb{Z}}$ be a lattice of dimension m with positive definite $Q_M\in\mathrm{Sym}^2(M^*_{\mathbb{Q}})$. Let D'' be the product of the primes p such that $M_{\mathbb{Z}_p}\not\subseteq M^*_{\mathbb{Z}_p}$ or $M^*_{\mathbb{Z}_p}/M_{\mathbb{Z}_p}$ is not cyclic. Assume

- $l-m\geq 4$, or
- $l=4$, $m=1$, *and $L_{\mathbb{Q}}$ has Witt rank* 1.

Then we have for each $\kappa\in\mathbb{Z}[(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})\otimes M^*_{\mathbb{Z}}]$:

(3) $\mathrm{vol}_{\mathcal{E}}\big(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;K)\big)=4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;0)$,

(4) $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}\big(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;K)\big)\equiv\frac{\mathrm{d}}{\mathrm{d}s}4\big(\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;s)\big)\big|_{s=0}$
in $\mathbb{R}_{2DD''}$.

Here $\mathbb{R}_N$ is $\mathbb{R}$ modulo rational multiples of $\log(p)$ for $p\mid N$, and the $\tilde{\lambda}$ and $\widetilde{\mu}$ are functions in $s\in\mathbb{C}$, given by certain Euler products (3.2.12) associated with representation densities of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$. The function $\widetilde{\mu}$ appears as the "holomorphic part" of a Fourier coefficient of the standard Eisenstein series associated with the Weil representation of L. Moreover K is the discriminant kernel and $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}}))$ is any toroidal compactification of the orthogonal Shimura variety (see below). Finally $\overline{\mathcal{E}}$ is the integral tautological bundle on the compact dual equipped with an equivariant metric on the restriction of its complex fibre to $\mathbb{D}$; the geometric/arithmetic volumes are computed w.r.t. the associated arithmetic automorphic line bundle (Definition 2.6.4) $\Xi^*\overline{\mathcal{E}}$ on $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}}))$.

In view of the Main Theorem it seems plausible that, if $L^*_{\mathbb{Z}}/L_{\mathbb{Z}}$ is cyclic, the function $4\tilde{\lambda}^{-1}(L;s)$ is always the correct normalizing factor in (K2).[4] This is in accordance with the observations of Kudla–Rapoport–Yang [42] in the case of Shimura curves.

[4]Of course this is only a statement about its first 2 Taylor coefficients.

For a more detailed introduction to the method of proof, we refer the reader to Section 1.4. An overview on known results in the direction of the conjectures will be given in Section 1.3.

In Chapter 2 up to Section 2.5 we recall the functorial theory of

- canonical integral models of toroidal compactifications of mixed Shimura varieties of Abelian type,
- their arithmetic automorphic vector bundles,

developed in the thesis of the author [25]. This theory, for general Hodge or even Abelian type, relies on an assumption regarding the stratification of the compactification (2.5.6) which was recently proven by Madapusi [49] and previously for the orthogonal Shimura varieties—spin-version—for $l \leq 5$ by Lan [46], (the Shimura varieties are of P.E.L. type in that case). The theory is crucial even for the precise formulation of our main result mentioned above. The reader is assumed to be familiar with the theory of rational Shimura varieties as developed by Deligne [16, 17] and to some extent with Pink's thesis [58] which extends the theory to the mixed case and contains the construction of rational canonical models of toroidal compactifications.

Our models are constructed locally (i.e. over an extension of $\mathbb{Z}_{(p)}$). Input data for the theory are p-integral mixed Shimura data (abbreviated p-MSD) $\mathbf{X} = (P_\mathbf{X}, \mathbb{D}_\mathbf{X}, h_\mathbf{X})$ consisting of an affine group scheme $P_\mathbf{X}$ over $\operatorname{Spec}(\mathbb{Z}_{(p)})$ of a certain rigid type (P) (see 2.1.6) and a set $\mathbb{D}_\mathbf{X}$ which comes equipped with a finite covering $h_\mathbf{X}\colon \mathbb{D}_\mathbf{X} \to \operatorname{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{X},\mathbb{C}})$ onto a certain conjugacy class, subject to some axioms, which are roughly Pink's mixed extension [58] of Deligne's axioms for a pure Shimura datum. Consider a compact open subgroup $K \subseteq P_\mathbf{X}(\mathbb{A}^{(\infty)})$ of the form $K^{(p)} \times P_\mathbf{X}(\mathbb{Z}_p)$ for a compact open subgroup $K^{(p)} \subseteq P_\mathbf{X}(\mathbb{A}^{(\infty,p)})$ (we call those admissible). For the toroidal compactification a certain rational polyhedral cone decomposition Δ of a natural conical complex $C_\mathbf{X}$ associated with $\mathbf{X}$ is needed. We call the collection ${}^K\mathbf{X}$ (respectively ${}^K_\Delta\mathbf{X}$) extended (compactified) p-integral mixed Shimura data (abbreviated p-EMSD, respectively p-ECMSD). These form categories where the morphisms ${}^K_\Delta\mathbf{X} \to {}^{K'}_{\Delta'}\mathbf{Y}$ are pairs (α, ρ) of a morphism α of Shimura data and $\rho \in P_\mathbf{Y}(\mathbb{A}^{(\infty,p)})$ satisfying compatibility conditions with the K's and Δ's. The construction of the models defines a functor M from p-ECMSD to the category of Deligne–Mumford stacks over reflex rings (above $\mathbb{Z}_{(p)}$). The functor, base-changed to $\mathbb{C}$ and restricted to p-EMSD, becomes naturally isomorphic to the one given by the construction of the analytic mixed Shimura variety. It is characterized uniquely by: Deligne's canonical model condition; Milne's extension property (integral canonicity); a stratification of the boundary into mixed Shimura varieties, together with boundary isomorphisms of the formal completions along these strata with similar completions of other (more mixed) Shimura varieties. These boundary isomorphisms, for the case of the symplectic Shimura varieties, are given by Mumford's construction [19, Appendix]. There is also a functor $\mathrm{M}^\vee$ ('compact' dual) from p-MSD to the category of schemes over reflex rings. The duals come equipped with an action of the group scheme $P_\mathbf{X}$, and we have morphisms of Artin stacks

$$\Xi_\mathbf{X}\colon\ \mathrm{M}({}^K_\Delta\mathbf{X}) \to [\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X},\mathcal{O}_\mathbf{X}}].$$

Those constitute a pseudonatural transformation of functors, are a model of the usual construction over $\mathbb{C}$ if Δ is trivial, and are compatible with the boundary isomorphisms. This is the theory of integral automorphic vector bundles. For more information on the 'philosophy' of these objects in terms of motives see Section 1.5.

In particular this enables us to do the following construction: Let $\mathbf{X}$ be, for simplicity, a pure Shimura datum with reflex field $\mathbb{Q}$, and let $\mathcal{E}$ be a $P_{\mathbf{X}}$-equivariant vector bundle on $\mathrm{M}^{\vee}(\mathbf{X})$ equipped with a $P_{\mathbf{X},\mathbb{R}}$-equivariant Hermitian metric $h_{\mathcal{E}}$ on its restriction to the image of the Borel embedding $h_{\mathbf{X}}(\mathbb{D}_{\mathbf{X}}) \hookrightarrow \mathrm{M}^{\vee}(\mathbf{X})(\mathbb{C})$. The above morphism of stacks and its 2-isomorphism over $\mathbb{C}$ with the analytic period construction allows us to obtain a *well-defined* Arakelov vector bundle $(\Xi^*\mathcal{E}, \Xi^*h_{\mathcal{E}})$. The metric $\Xi^*h_{\mathcal{E}}$, however, has (rather mild) singularities along the boundaries of toroidal compactifications (see below). Without the existence of this canonical pullback, it would not even make sense to speak of arithmetic volumes.

The remaining sections of Chapter 2 are completely revised with respect to [25]: In Section 2.6 we define integral Hermitian automorphic vector bundles and the notions of arithmetic and geometric volume. Furthermore we set up an Arakelov theory which has enough properties to deal with singularities of the natural Hermitian metrics on "fully decomposed" automorphic vector bundles. This uses work of Burgos, Kramer, and Kühn [14, 15] but the arithmetic Chow groups are defined using a technically simpler method.

In Sections 2.7 and 2.8 a precise general q-expansion principle is derived from the abstract properties of Section 2.5.

Chapter 3 is concerned entirely with the theory of orthogonal Shimura varieties: In Section 3.1 the structure of the models of orthogonal Shimura varieties is investigated and special cycles are defined.

In Section 3.2 we use the general q-expansion principle to prove that Borcherds products with their natural norm yield *integral* sections of an appropriate integral Hermitian line bundle. Among other things the product expansions of Borcherds' are adelized and their Galois properties investigated.

In Section 3.3 we prove that the bundle of vector valued modular forms for the Weil representation (which appear as input forms in the construction of Borcherds products) has a rational structure. Then we use this to construct input forms with special properties which will be needed later.

In Section 3.4 we relate Borcherds' theory and Arakelov geometry on the orthogonal Shimura varieties. Its main result is an (averaged) arithmetic Siegel–Weil formula which will be crucial for the proof of the Main Theorem.

In Section 3.5 the Main Theorem is proven. An overview on the proof may be found in Section 1.4.

In Appendix A additional material on quadratic forms, on "lacunarity of modular forms," and on semilinear representations is provided which will be needed in the proofs in Section 3.5. Appendix B contains a continuation of the calculation in [26] of the function $\lambda(L_{\mathbb{Z}}; s)$ for the special case of lattices with square-free discriminant.

Finally, it is a pleasure to thank the Department of Mathematics and Statistics at McGill University and especially Henri Darmon, Jayce Getz and Eyal Goren, for providing a very inspiring working environment during the preparation of this book.

1.2. A brief introduction to Siegel–Weil theory

1.2.1. Consider two lattices $L_{\mathbb{Z}} \cong \mathbb{Z}^l$, and $M_{\mathbb{Z}} \cong \mathbb{Z}^m$ with integral and positive definite quadratic forms Q_L, and Q_M. It is a classical problem, to which already Gauss, Euler and in particular Siegel devoted themselves, to determine the representation number, that is, the number of elements in the set of isometric embeddings

$$\mathrm{I}(L_{\mathbb{Z}}, M_{\mathbb{Z}}) = \{\alpha\colon M_{\mathbb{Z}} \hookrightarrow L_{\mathbb{Z}} \mid \alpha \text{ is an isometry}\}.$$

It includes (for $m = 1$) questions like: "In how many ways can an integer be represented as a sum of l squares?".

If $M_{\mathbb{Z}} = \mathbb{Z}^m$ then Q_M is given by an element in $\mathrm{Sym}^2\big((\mathbb{Z}^m)^*\big)$. We write $\langle Q\rangle$ for the lattice $\mathbb{Z}^m$ with quadratic form given by Q. The *generating series*, the *theta series* of $L_{\mathbb{Z}}$,

$$\Theta_m(L_{\mathbb{Z}}; \tau) = \sum_{Q\in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} \# \mathrm{I}(L_{\mathbb{Z}}, \langle Q\rangle) \exp(2\pi i Q\cdot\tau), \tag{1.3}$$

(here τ is an element in Siegel's upper half space $\mathbb{H}_m \subset (\mathbb{C}^m \otimes \mathbb{C}^m)^s$, the subset of elements with positive definite imaginary part) is a *Siegel modular form* of weight $l/2$ for a certain congruence subgroup of Sp'_{2m} (the symplectic or metaplectic group, according to the parity of l). For example $\Theta_1(\langle 1\rangle; \tau)$ is just the classical theta function.

Under some conditions on the dimensions, a certain weighted sum of these theta functions over all *classes* $L_{\mathbb{Z}}^{(i)}$ in the *genus* $L_{\hat{\mathbb{Z}}}$ is an Eisenstein series (cf. [26, Section 4] for details):

1.2.2. Theorem (Siegel–Weil). *If* $l > 2m+2$, *we have*

$$\sum_i c_i \Theta_m(L_{\mathbb{Z}}^{(i)}; \tau) = E_m(\Phi; \tau, s_0).$$

The additional parameter s_0 indicates that this Eisenstein series is in fact the (holomorphic) special value of a nonholomorphic Eisenstein series $E_m(\Phi;\tau,s)$ at $s = s_0 := (l-m+1)/2$. The Fourier coefficients of the series are given by a product formula

$$\mu(L_{\mathbb{Z}}, \langle Q\rangle; s, y) = \mu_\infty(L, \langle Q\rangle; s, y) \prod_p \mu_p(L_{\mathbb{Z}_p}, \langle Q\rangle; s). \tag{1.4}$$

Here y is the imaginary part of τ—its appearance indicates that this series is non-holomorphic for general s. At $s = 0$ the μ_p are just the p-adic volumes of the 'spheres' $I(L, \langle Q\rangle)(\mathbb{Z}_p)$, classically called *representation densities*. For almost all p, the functions μ_p are very simple polynomials in p^{-s} (see, e.g., [26, Theorem 8.1]). Otherwise they may be computed by determining sufficiently many representation numbers of the congruences modulo p^n.

Essentially, the Siegel–Weil formula (1.2.2) is valid, if and only if $l > m+1$, but if $l \leq 2m+2$, the value of the Eisenstein series has to be defined via analytic continuation in s and the theta function has sometimes to be complemented by indefinite coefficients.

The mere fact that the representation numbers (in an average over classes) should be given by a product over local volumes or densities can be explained easily in the adelic language:

1.2.3. Let $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ now be an integral lattice in an arbitrary quadratic space (not necessarily definite) and $M_{\mathbb{Q}}$ a positive definite quadratic space. Assume $l - m \geq 3$, for simplicity, for the rest of the discussion. On the adelic points $\mathrm{SO}(L_{\mathbb{A}})$ of the special orthogonal group of $L_{\mathbb{Q}}$, there is a canonical measure μ. It is a product over local measures μ_ν on the various $\mathrm{SO}(L_{\mathbb{Q}_\nu})$, constructed by any algebraic volume form defined over $\mathbb{Q}$ [65]. The product μ is independent of the choice of this form. The volume of $\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})$ which turns out to be finite, is called the **Tamagawa number**, and we have

1.2.4. Theorem ([65]). *For $l \geq 3$*

$$\mathrm{vol}\big(\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})\big) = 2.$$

From this the fact that the representation numbers (in an average over classes) should be given by a product over local volumes already follows, as we will explain now (in a slightly broader context):

1.2.5. Let $\varphi \in S(L_{\mathbb{A}^{(\infty)}} \otimes M^*_{\mathbb{A}^{(\infty)}})$ be a Schwartz-Bruhat function (i.e., locally constant with compact support). Let $K = \prod_p K_p$ be a compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ which stabilizes φ. For example, K could be the stabilizer of the lattice $L_{\widehat{\mathbb{Z}}}$ and φ the characteristic function of $L_{\widehat{\mathbb{Z}}}$. Let K_∞ be a maximal compact subgroup of $\mathrm{SO}(L_{\mathbb{R}})$. (If L is definite, this will be equal to $\mathrm{SO}(L_{\mathbb{R}})$.)

From 1.2.4 we may infer that the volume of the real analytic orbifold

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash(\mathrm{SO}(L_{\mathbb{A}})/K_\infty K)],$$

induced by the quotient of μ_∞ and some measure on K_∞, is:

$$2\prod_\nu \mathrm{vol}_\nu^{-1}(K_\nu). \tag{1.5}$$

We have a finite disjoint decomposition

$$\mathrm{I}(L,M)(\mathbb{A}^{(\infty)}) \cap \mathrm{supp}(\varphi) = \bigcup_i K\alpha_i$$

If this set is nonempty, we have by Hasse's principle an $\alpha' \in \mathrm{I}(L,M)(\mathbb{Q})$ and hence $g_i \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ with $g_i^{-1}\alpha' = \alpha_i$. There is a lattice $L_{\mathbb{Z}}^{(i)} \subset L_{\mathbb{Q}}$ satisfying $L_{\widehat{\mathbb{Z}}}^{(i)} = g_i^{-1}L_{\widehat{\mathbb{Z}}}$. We denote by abuse of notation by $\alpha_i^\perp$ the lattice $\mathrm{im}(\alpha')^\perp \cap L_{\mathbb{Z}}^{(i)}$. We have $\alpha_i^\perp \otimes \widehat{\mathbb{Z}} \cong \mathrm{im}(\alpha_i)^\perp$. However, only the genus of $\alpha_i^\perp$ is well-defined, but we will use the notation only for objects which depend only on this genus.

Consider the symmetric space[5]

$$\mathbb{D}(L) = \{\text{maximal negative definite subspaces of } L_{\mathbb{R}}\} = \mathrm{SO}(L_{\mathbb{R}})/K_\infty.$$

We have an embedding $\mathbb{D}(\alpha_i^\perp) \times \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big) \hookrightarrow \mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$, given by the natural inclusion of $\mathbb{D}(\alpha_i^\perp) \hookrightarrow \mathbb{D}(L)$ and multiplication of the adelic part by g_i from the right.

We form the *special cycle* $\mathrm{Z}(L,M,\varphi;K)$, the following formal sum (with real coefficients):

$$\sum_i \varphi(\alpha_i)\Big[\mathrm{SO}((\alpha_i^\perp)_{\mathbb{Q}})\backslash\mathbb{D}(\alpha_i^\perp) \times \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big)\Big/\Big({}^{g_i}K \cap \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big)\Big)\Big]$$

---

[5] *Caution*: This definition differs from the later definition of $\mathbb{D}_{\mathbf{O}(L)}$ in case that L has signature $(l-2,2)$

which we consider, by means of the embeddings above, as a formal sum of real analytic suborbifolds of $[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]$. It does not depend on the choices made above.

The *canonical* measures [26, 2.4] on $\mathrm{SO}(L_{\mathbb{Q}})$, $\mathrm{SO}(\alpha_i^{\perp})$ and $\mathrm{I}(L,M)$ over any $\mathbb{Q}_\nu$ are related by an orbit equation [26, 5.6]—an equation of the shape:

$$\text{'volume of space'} = \sum_{\text{orbits}} \frac{\text{'volume of group'}}{\text{'volume of stabilizer'}},$$

similar to the corresponding formula for actions of finite groups on sets.

From this and (1.5) above

$$\frac{\mathrm{vol}(Z(L,M,\varphi;K))}{\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)} = \frac{\mathrm{vol}(K_\infty)}{\mathrm{vol}(K'_\infty)} \int_{\mathrm{I}(L,M)(\mathbb{A}^{(\infty)})} \varphi(\alpha)\mu(\alpha) \tag{1.6}$$

follows immediately. K'_∞ is any maximal compact subgroup of any of the $\mathrm{SO}(\alpha_{i,\mathbb{R}}^{\perp})$. We define $\mu_\infty(L,M)$ to be the quantity $\mathrm{vol}(K_\infty)/\mathrm{vol}(K'_\infty)$ (computed w.r.t. the canonical measures). If L is definite, it is equal to:

$$\mathrm{vol}(\mathrm{I}(L,M)(\mathbb{R})) = \prod_{k=l-m+1}^{l} 2\frac{\pi^{k/2}}{\Gamma(k/2)}.$$

Observe that

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)] = \bigcup_j [(\mathrm{SO}(L_{\mathbb{Q}}) \cap {}^{g_j}K)\backslash\mathbb{D}(L)],$$

with respect to a set $\{g_j\}_j$ of representatives of $\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K$, i.e., of the *classes of* $\mathrm{SO}(L_{\mathbb{Q}})$ with respect to the compact open group K. (If K is the stabilizer of a lattice $L_{\hat{\mathbb{Z}}}$, this coincides with the classical notion of classes in the genus $L_{\hat{\mathbb{Z}}}$.) Similarly, we have

$$\mathrm{Z}(L,M,\varphi;K) = \sum_{i,k} \varphi(\alpha_{ik})\big[\big(\mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{Q}}\big) \cap K^{g_{ik}}\big)\backslash\mathbb{D}(\alpha_i^{\perp})\big], \tag{1.7}$$

where $\{g_{ik}\}_k$ is a set of representatives of the classes of $\mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{Q}}\big)$ w.r.t. ${}^{g_i}K \cap \mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{A}^{(\infty)}}\big)$.

Let now K be the stabilizer of $L_{\hat{\mathbb{Z}}}$ and φ the characteristic function. We have the following easy

1.2.6. Lemma. *There is a bijection*

$$\left\{\begin{matrix}\textit{class } L_{\mathbb{Z}}^{(j)} \textit{ in the genus } L_{\hat{\mathbb{Z}}}, \\ \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\textit{-orbit } \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \textit{ in } \mathrm{I}(L^{(j)},M)(\mathbb{Z})\end{matrix}\right\}$$
$$\xrightarrow{\sim} \left\{\begin{matrix}\mathrm{SO}(L_{\hat{\mathbb{Z}}})\textit{-orbit } \mathrm{SO}(L_{\hat{\mathbb{Z}}})\alpha \textit{ in } \mathrm{I}(L,M)(\widehat{\mathbb{Z}}), \\ \textit{class in } \mathrm{SO}(\alpha_{\mathbb{Q}}^{\perp})\backslash \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp})/K \cap \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp})\end{matrix}\right\}.$$

We have, of course, a similar statement for any K.

We denote the cycle in this case by $\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}})$ and it is, according to the lemma and (1.7), equal to:

$$\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}) = \sum_j \sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \subset \mathrm{I}(L^{(j)},M)(\mathbb{Z})} \big[\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)\backslash\mathbb{D}(L)\big].$$

1.2.7. Now, if the form Q_L is *positive definite*, the quotient of volumes (1.6) has an interpretation as a global representation number. For this observe that in this case

$$\mathrm{vol}\big(\mathrm{SO}(L_{\mathbb{Z}})\backslash\mathbb{D}(L)\big) = \frac{1}{\#\,\mathrm{SO}(L_{\mathbb{Z}})}$$

and similarly

$$\mathrm{vol}\big((\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)}))\backslash\mathbb{D}(\alpha^{\perp})\big) = \frac{1}{\#\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)}.$$

Furthermore, we have by the set theoretical orbit equation,

$$\frac{\#\,\mathrm{I}(L^{(j)}, M)(\mathbb{Z})}{\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})} = \sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \in \mathrm{I}(L^{(j)}, M)(\mathbb{Z})} \frac{1}{\#\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)}.$$

Hence we get

$$\frac{\mathrm{vol}\big(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}})\big)}{\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)} = \frac{\sum_j \#\,\mathrm{I}(L_{\mathbb{Z}}^{(j)}, M)(\mathbb{Z})/\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}{\sum_j 1/\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}$$

which is precisely a weighted sum over the representation numbers. Combined with (1.6), we get *Siegel's formula*. The deep part, of course, is hidden in Theorem 1.2.4.

1.2.8. If the quadratic form on L is *indefinite*, say of signature (p, q), then these representation numbers do not make sense because there are always infinitely many isometries. However, equation (1.6) tells us, what the correct analogue in the indefinite case is: the quotient of volumes

$$\frac{\mathrm{vol}\big(\mathrm{Z}(L, M, \varphi; K)\big)}{\mathrm{vol}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)])}.$$

For every cohomology theory H (in a very broad sense) one might in addition consider the *classes* $[Z(L, \langle Q\rangle, \varphi; K)]^H$ of these cycles and define their generating theta series:

$$\Theta_m^H(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q\rangle, \varphi; K)]^H \cup e_q^{m-r(Q)} \exp(2\pi i Q \cdot \tau),$$

where e_q is a certain Euler class, and $r(Q)$ is the rank of Q. One always expects modularity of this function and a relation to Eisenstein series.

Kudla and Millson [37, 38] have shown (generalizing work of Hirzebruch and Zagier [24]) that the generating series

$$\Theta_m^{\mathrm{B}}(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q\rangle, \varphi; K)]^{\mathrm{B}} \cup e_q^{m-r(Q)} \exp(2\pi i Q \cdot \tau),$$

with values in the Betti cohomology groups

$$H^{(p-m)q}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)], \mathbb{C})$$

is a modular form. Under certain conditions on l, m and the Witt rank of L, its 'degree' is the special value of an Eisenstein series:

$$\langle \Theta_m^{\mathrm{B}}(L, \varphi; \tau), e_q^{l-m}\rangle = \mathrm{vol}_{e_q}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]) E_m(\Phi; \tau, s_0).$$

The latter equation follows essentially again from the Siegel–Weil formula (in its full generality) or the Tamagawa number result, respectively. If $L_{\mathbb{Q}}$ is anisotropic, the locally symmetric space is compact and the pairing on the left is the degree of