WHEN LIFE IS LINEAR

FROM COMPUTER GRAPHICS TO BRACKETOLOGY

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When Life is Linear

From Computer Graphics to Bracketology

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MAA

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To my parents, Jan and Myron,
thank you for your support,
commitment, and sacrifice
in the many nonlinear
stages of my life

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and edX. The book and online course are not dependent on each other. Participants of the MOOC do not need this book, and readers of this book do not need to participate in the MOOC. The experiences are distinct, but for those interested, I believe the book and online course will complement each other. One benefit of the MOOC is a collection of online resources being developed. Many of the ideas in this book require a computer and either mathematical software or a computer program to calculate the underlying mathematics. From graphics that have thousands of pixels to datasets that have hundreds or thousands of entries, computing can be necessary even for ideas that can be easily demonstrated by hand on small datasets. I look forward to seeing the ideas of this book shared, explored, and extended with the online community.

The book has one more goal. A few years ago, a group of students gathered for a recreational game of Math Jeopardy. One of the subject areas was linear algebra, which turned out to be the hardest for the students. As the students struggled to recall the content necessary to answer the questions, one student joked, "Well, at least this stuff isn't useful. Maybe that's why I can't remember it." Even looking at the table of contents of this book, you'll see why I see linear algebra as very useful. I tend to believe that at any second in any day linear algebra is being used somewhere in a real world application. My hope is that after reading this book more people will say, "I really need to remember these ideas. They are so useful!" If that person is you, maybe you'd begin sharing the many ideas you created and explored as you read this book. I wish you well as you discover matrices awaiting to be formed.

Acknowledgments

First and foremost, I'd like to thank Tanya Chartier, who travels this journey of life with me as my closest companion. Our paths, while not always linear, have always felt parallel. Thanks to Karen Saxe, chair of the Anneli Lax New Mathematics Library Editorial Board, for her encouragement, many insights, and commitment to this project. Along with Karen, I thank the Anneli Lax New Mathematics Library Editorial Board for their reading of the book and insightful edits that improved the book and its content. I want to thank Michael Pearson, Executive Director of the Mathematical Association of America, for his support for this project and its connection to the MOOC. I'd also like to thank Shane Macnarmara for diving into the development of the online resources that accompany the MOOC and, as such, this book as well. Shane's commitment enhanced the online resources and will benefit many people who use them. Finally, I want to thank the folks involved in the Davidson College MOOC, Allison Dulin, Kristen Eshleman, and Robert McSwain, for the many ways their work on the online course enhanced the content in these printed pages. I am grateful to my colleagues, near and far, who support my work and to the many students who sit and chat about such things and offer insightful ideas that enhance and enrich my activities. It is impossible to thank everyone who is involved in a project and supports its content. For those unmentioned, may you see your influence in these pages and enjoy the way our interaction materialized into printed work.

Preface

In the 1999 film *The Matrix*, there is the following interchange between Neo and Trinity.

Neo: *What is the Matrix?*
Trinity: *The answer is out there, Neo, and it's looking for you, and it will find you if you want it to.*

As an applied mathematician specializing in linear algebra, I see many applications of linear systems in the world. From computer graphics to data analytics, linear algebra is useful and powerful. So, to me, a matrix, connected to an application, is indeed out there–waiting, in a certain sense, to be formed. When you find it, you can create compelling computer graphics or gain insight on real world phenomenon.

This book is meant to engage high schoolers through professors of mathematics in applications of linear algebra. The book teaches enough linear algebra to dive into applications. If someone wanted to know more, a Google search will turn up a lot of information. If more information is desired, a course or textbook in linear algebra would be the next step. This book can either serve as a fun way to step into linear algebra or as a supplementary resource for a class.

My hope is that this book will ignite ideas and fuel innovation in the reader. From trying other datasets for data analytics to variations on the ideas in computer graphics, there is plenty of room for your personal discovery. So, this book is a stepping stone, intended to springboard you into exploring your own ideas.

This book is connected to a Massive Open Online Course (also known as a MOOC) that will launch in February of 2015 through Davidson College

Contents

1
X Marks the Spot

In this book, we will think linearly. In two dimensions, this means a line. In three dimensions, we're talking about a plane. In higher dimensions, math helps us work with hyperplanes. A lot of the world isn't linear. The world, especially the natural world, often offers beautiful curves. Yet, like the horizon, curvature, if viewed in the right way, can look linear.

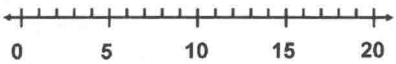
The ability to approximate curves with lines will be important to many portions of this book. To get a visual sense of modeling curved space with lines, consider sketching only with straight lines. How about drawing a portrait? I'll lay down a series of dots that approximate an image and then I'll draw one continuous line through all the points. I'll start and end at the same point. See Figure 1.1 for an example. Recognize the image in Figure 1.1? Such visual art, called TSP Art, was introduced and developed in [3, 4]. "TSP" stands for "traveling salesperson problem" since the underlying dots can be viewed as cities and the line segments between dots indicate the route the salesperson will make through the cities. Such problems help minimize travel.

Though the image is not the original portrait, the drawing is recognizable. The line drawing captures, in this case, visual components of the original image. Later, we will use linear phenomenon to model sports performance enabling us to predict future play.

But, let's not get too far ahead of ourselves. Most of this book will explore linear systems, essentially puzzles written as equations. Let's see an example that I'll pose in the form of a magic trick.

Figure 1.1. A portrait of Marilyn Monroe drawn with straight lines. Photograph courtesy of Robert Bosch.

Think of a number between 1 and 20. Double the number. Now, add 8. Next, divide by 2. Subtract your original number and mark the spot on the number line where this computed number lies with an x.



I bet I know where x lies on the number line. It lies at the value 4. How did I know? I did this by first using a variable y, and y equals the original number you chose, regardless of its value. In the second step, you doubled it producing $2y$. Adding 8 resulted in $2y + 8$, and then dividing by 2 led to $(2y + 8)/2 = y + 4$. Finally, I asked you to subtract that original number. I never knew its value but did know that it equalled y so, when you subtract y from $y + 4$, you indeed get 4. So, my guess wasn't much of a guess; rather it was some algebra—in fact, *linear* algebra since all the terms (y, $2y$, $2y + 8$ and so forth) are linear. I never needed to know your original number. In fact, it wouldn't have mattered if you selected a number between -100 and 100 or a number with a million digits. Regardless, your final value would always result in 4.

Solving for unknowns is a powerful tool of linear algebra. Here's a problem you may recall from a math class that again looks for one unknown number.

> The Alpha Train, traveling 80 miles per hour (mph), leaves Westville heading toward Eastburg, 290 miles away. Simultaneously,

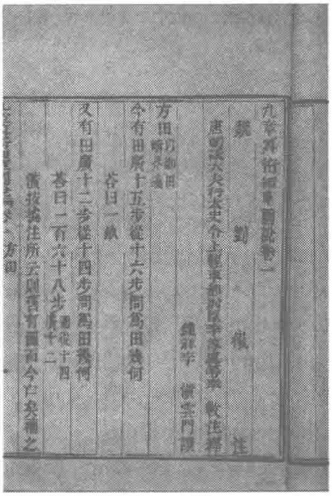

魏 劉 徽 注
方田 以御田疇界域
今有田廣十五步從十六步問為田幾何
荅曰一畝
又有田廣十二步從十四步問為田幾何
荅曰一百六十八步 圖從十四廣十二

Figure 1.2. The *Jiuzhang Suanshu*, a Chinese manuscript dating from approximately 200 BC. Pictured is the opening of Chapter 1.

> the Beta Train, leaves Eastburg, traveling 65 mph, toward Westville. When do the two trains meet?

Here x will again denote the unknown. In this case, we translate the word problem; $80x + 65x = 290$. With that, we solve and find $x = 2$. In two hours, the trains will meet when the total of their distances equals the distance between the cities.

Puzzles like this date back thousands of years. Here's one from a Chinese manuscript from approximately 200 BC called the *Jiuzhang Suanshu*, which contained 246 problems intended to illustrate methods of solution for everyday problems in areas such as engineering, surveying, and trade. [12]

> There are three classes of grain, of which three bundles of the first class, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?

This problem involves three unknowns rather than one as in our problem with the train.

Let x denote the number of measures of the first class grain in one bundle. Let y and z equal the number of measures of the second and third