Kerstin Dächert

Adaptive Parametric Scalarizations in Multicriteria Optimization

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Contents

1 Introduction

1.1 Multiobjective Optimization

Over one hundred years ago, Francis Edgeworth (1845–1926) and Vilfredo Pareto (1848–1923) laid the foundations of what is today called multicriteria decision making. The basic assumption of multicriteria decision making is that whenever a decision has to be taken, not only one but multiple objectives have to be taken into account. Moreover, in general, these objectives are competing, i.e., no solution or decision action exists for which all objectives can be met best simultaneously. An example is given by different products in a market. Since, in general, a cheap product has a rather bad quality while a product of good quality is rather expensive, a compromise between the objectives 'price' and 'quality' has to be found. We can only exclude products from consideration that are at least as expensive and, simultaneously, of at most the same quality as some other product. This is the basic idea of dominance and nondominance in multiobjective optimization: Of interest are exactly those solutions (products) that cannot be improved with respect to one criterion without being impaired with respect to at least one other criterion. In the literature, these solutions are called 'nondominated', 'efficient' or, in honor to the fathers of multicriteria decision making, 'Pareto' or sometimes also 'Edgeworth-Pareto optimal', see Section 2.1 for a precise definition. Due to the conflicting nature of the objectives, there is, in general, not only one but a set of Pareto optimal solutions.

As already suggested by the titles of the early publications *Mathematical psychics* (Edgeworth, 1881) and *Cours d'Economie Politique* (Pareto, 1896), multicriteria decision making is an interdisciplinary field that, from the very beginning up to today, attracts researchers and practitioners from various disciplines as economics, psychology, mathematics and computer as well as engineering science. Thereby, the interests range from the theoretical analysis of multiobjective optimization problems over the practical computation and representation of solutions up to economical utility theory and questions of human behavior in decision making. In brief, this thesis presents new theoretical results for generating Pareto optimal solutions and shows the practical usefulness of the new theory.

1.2 Outline of This Thesis

The content of this thesis is organized in two parts and nine chapters. The first three chapters present the basics. The fourth and fifth chapter, which build Part I, contain new theoretical results. Their practical application is demonstrated in Part II, which consists of chapters six to eight. The last chapter summarizes the results of this thesis. In what follows we describe the content of each chapter in more detail.

Chapter 2 assembles the relevant definitions, notions and concepts from the literature that are needed in the following. First, we provide general definitions from the field of multicriteria optimization. After that we introduce the notions of representations and approximations of the nondominated set and indicate quality criteria from the literature. Then scalarizations as a well known concept to solve multicriteria optimization problems are presented. Finally, the idea of a parametric algorithm that consists in the iterative solution of scalarizations with varying parameters is specified and the notions of a priori and adaptive (a posteriori) parameter schemes are introduced.

Chapter 3 provides a detailed literature review on methods using (adaptive) parametric algorithms. The survey on this topic starts with early publications dating from the sixties of the last century and ends with very recent publications. As several methods are solely applicable to the bicriteria case, we organize the literature review into two sections, one devoted particularly to the bicriteria and the other one to the general multicriteria case.

After the introduction, the preparation of the basics and the presentation of related literature, new theoretical results are presented in **Part I**. In brief, Chapter 4 deals with new adaptive parameter schemes for well-known scalarization methods with augmentation terms, particularly the augmented weighted Tchebycheff method. Chapter 5 is concerned with the general framework of a new parametric algorithm.

In Chapter 4 we derive an adaptive parameter scheme for the augmented weighted Tchebycheff method which is the first classic scalarization for which an augmentation term has been introduced. So far, only the weights have been controlled in an adaptive way but the augmentation parameter has been chosen fixed to a small positive constant. As reported in the literature, on the one hand, numerical issues arise when this constant is too small and, on the other hand, nondominated points are missed when the constant is chosen too large. We construct all parameters of the augmented weighted Tchebycheff method in an adaptive way such that every nondominated point of a discrete multicriteria optimization problem can be generated and, at the same time, the augmentation parameter can be chosen as large as possible up to a feasible upper bound. Besides the classic augmented weighted Tchebycheff method we consider a generalized problem formulation that contains an augmentation parameter for each objective and, thus, provides more flexibility. The generalized formulation is particularly useful for the application to continuous problems, as it allows to incorporate a given trade-off among the objectives. For bicriteria problems it is well known that a prescribed two-sided trade-off can be translated into suitable parameters of a generalized augmented weighted Tchebycheff problem. We improve existing approaches by proposing an adaptive parameter scheme that takes all parameters, i.e., also the weights, into account. Finally, augmented variants of the ε-constraint method from the literature are discussed. We show that the augmentation parameter of an augmented ε-constraint scalarization can be determined in the same way as it is proposed for the augmented weighted Tchebycheff method.

In Chapter 5 we develop the general framework of an adaptive parametric algorithm that is based on a systematic decomposition of the search region, i.e., the region potentially containing further nondominated points. We particularly study the number of subproblems that have to be solved to generate complete representations for discrete problems. In the literature, the best known upper bound on the number of subproblems in the tricriteria case depends quadratically on the number of nondominated points. By indicating a new parametric algorithm in which at most three subproblems are solved per nondominated point, we improve the former quadratic to a linear upper bound. Thereby, the main key is a new decomposition criterion which avoids redundancy. The parametric algorithm can be applied with any scalarization that is suited for non-convex or discrete problems. If the ε-constraint method is used, we can reduce the upper bound further and show that at most two subproblems per nondominated point are sufficient to obtain a complete representation. Finally, we propose an extension of the new algorithm for any number of objectives.

The theoretical results of Part I are validated computationally in **Part II**. Thereby, the results of Part I are combined in the sense that the adaptive parameter scheme from Chapter 4 is employed for each subproblem that is solved in the parametric algorithm derived in Chapter 5.

In Chapter 6 we generate complete representations for discrete problems. In the bicriteria case the performance of different variants of Tchebycheff scalarizations is examined. Besides the validation of the adaptive parameter scheme proposed in Chapter 4 we compare the adaptive parameter scheme to the classic fixed choice of the augmentation parameter which is common in the literature. In particular, we show computationally that already for small instances of knapsack problems nondominated points are missed with the classic approach but not with the adaptive parameter selection. We also study further algorithmic variants with local reference points by which larger values for the augmentation parameter can be obtained. In the tricriteria case we validate the formulas for the parameters of the augmented weighted Tchebycheff method as well as the upper bound on the number of subproblems derived in Chapter 5. In all instances the complete nondominated set is computed reliably with the help of the adaptive parameter scheme. Moreover, the predicted upper bound on the number of subproblems is met exactly in all instances. Besides the validation of our new parametric algorithm, we also compare it with three state of the art methods for the generation of complete representations. Our algorithm clearly outperforms one of the three algorithms and can compete with the other two in the sense that no algorithm outperforms the other with respect to the number of subproblems solved and the required computational time.

In Chapter 7 we apply the new adaptive parametric algorithm to continuous problems, for which incomplete representations of the nondominated set are sought. We use common quality criteria to measure the quality of the representations. In order to refine the representations iteratively, we propose different selection rules based on the volume of the boxes into which the search region is decomposed and the contribution to the dominated hypervolume. Tests with bi- and tricriteria problems from the literature are performed. We compare different variants of Tchebycheff methods employing an adaptive parameter scheme with an a priori ε-constraint method. We observe that with the adaptive methods considerably less infeasible or redundant subproblems are generated than with the a priori method, in general. The adaptive approaches perform particularly well when the nadir point is not known and its estimate is rather bad. Hence, they are particularly useful for problems with more than two criteria.

Chapter 8 treats a real-world problem in which the multicriteria control of sewer networks is considered. Within a preliminary offline analysis we aim at constructing a discrete representation of the nondominated set. Since the single-criterion solver used for the subproblems is interrupted before its termination, it typically does not provide local or global minima but intermediate solutions. These solutions often correspond to dominated or even infeasible points. Therefore, we can only construct a very scarce discrete approximation of the nondominated set. Moreover, due to numerical issues, an a priori parameter scheme yields better results than an adaptive scheme in some test cases. This shows that the performance of the underlying single-criterion solver is crucial for the successful use of adaptive parameter schemes. If the generated points are not nondominated or close to nondominated points, an a priori parameter selection might be preferred.

Chapter 9 contains a summary of the results of this thesis. Ideas for future research are indicated directly at the end of each chapter of Parts I and II.

2 Preliminaries

In this chapter we collect the relevant notions, definitions and concepts that are used in this thesis. They are common knowledge and can be found in textbooks on multicriteria optimization, e.g., in Chankong and Haimes (1983), Steuer (1986), Miettinen (1999), Jahn (2004) or Ehrgott (2005).

2.1 Terminology and Definitions

We consider multiple criteria optimization problems

$$\min_{x \in X} f(x) = (f_1(x), \ldots, f_m(x))^\top \tag{2.1}$$

with $m \geq 2$ *objective functions* $f_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, and with *feasible set* $X \subseteq \mathbb{R}^n$. We assume that the functions $f_i, i = 1, \ldots, m$, are continuous and that X is non-empty and compact. If X is a discrete finite set, we call Problem (2.1) discrete. The image of the feasible set X is denoted by $Z := f(X) \subseteq \mathbb{R}^m$ and is called *set of feasible outcomes*.

To simplify notation, we will often refer to the points in Z without relating them back to their preimages in the feasible set. Consequently, we equivalently formulate Problem (2.1) in the outcome space as

$$\min_{z \in Z} z = (z_1, \ldots, z_m)^\top. \tag{2.2}$$

For two vectors $z, \bar{z} \in Z$ we define

$$\begin{aligned} z < \bar{z} \quad &:\Leftrightarrow \quad z_i < \bar{z}_i \quad \forall i = 1, \ldots, m, \\ z \leq \bar{z} \quad &:\Leftrightarrow \quad z_i \leq \bar{z}_i \quad \forall i = 1, \ldots, m \text{ and } \exists j \in \{1, \ldots, m\} : z_j < \bar{z}_j, \\ z \leqq \bar{z} \quad &:\Leftrightarrow \quad z_i \leq \bar{z}_i \quad \forall i = 1, \ldots, m. \end{aligned} \tag{2.3}$$

The symbols $>$, $\geq$ and $\geqq$ are used accordingly. As there exists no canonical ordering on $\mathbb{R}^m$ for $m \geq 2$, a definition of optimality is required. We use the Pareto concept of optimality: A solution $\bar{x} \in X$ is called *Pareto optimal* or *efficient* if there does not exist a feasible solution $x \in X$ such that $f(x) \leq f(\bar{x})$. The corresponding objective vector $f(\bar{x}) \in \mathbb{R}^m$ is called *nondominated* in this case. If, on the other hand, $f(x) \leq f(\bar{x})$ for some feasible $x \in X$, we say that $f(x)$ *dominates* $f(\bar{x})$, and x *dominates* $\bar{x}$. If strict inequality holds for all m components, i.e., if $f(x) < f(\bar{x})$, then x *strictly dominates* $\bar{x}$. If there exists no feasible solution $x \in X$ that strictly dominates $\bar{x}$, then $\bar{x}$ is called *weakly Pareto optimal* or *weakly efficient*. We denote the set of efficient solutions of (2.1) by X_E and refer to it as the *efficient set*, i.e.,

$$X_E := \{x \in X : \nexists \tilde{x} \in X : f(\tilde{x}) \leq f(x)\}. \tag{2.4}$$

The image set of the set of efficient solutions is denoted by

$$Z_N := f(X_E) = \{z \in Z : \nexists \tilde{z} \in Z : \tilde{z} \leq z\} \tag{2.5}$$

and is called the *nondominated set* of problem (2.1). In general, one nondominated point $f(\bar{x})$ might have more than one preimage $\bar{x} \in X$. However, throughout this thesis, it is sufficient to know one efficient solution per nondominated point.

A point $\bar{x} \in X$ is called *properly efficient* according to Geoffrion (1968) if it is efficient and if there exists a scalar $M > 0$ such that for each $i = 1, \ldots, m$ and each $x \in X$ satisfying $f_i(x) < f_i(\bar{x})$ there exists an index $j \neq i$ with $f_j(x) > f_j(\bar{x})$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M. \tag{2.6}$$

An efficient point that is not properly efficient is called *improperly efficient*. Note that if the outcome space Z is discrete and finite, every efficient point is properly efficient.

The notion of *trade-off* is closely related to the definition of proper efficiency. According to Chankong and Haimes (1983), for given $x, \bar{x} \in X$, the ratio of change $T_{ij}(x, \bar{x})$ involving objective functions f_i and f_j, $i, j = 1, \ldots, m$, $i \neq j$, is defined as

$$T_{ij}(x, \bar{x}) := \frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \tag{2.7}$$

for $f_j(x) \neq f_j(\bar{x})$. Note that if $f_i(x) \neq f_i(\bar{x})$, then $T_{ij}(x, \bar{x}) = (T_{ji}(x, \bar{x}))^{-1}$ and $T_{ij}(x, \bar{x}) = T_{ij}(\bar{x}, x)$ hold. In Kaliszewski and Michalowski (1997), for $\bar{z} \in Z$ and a problem in maximization format, the trade-off $T_{ij}^G(\bar{z})$ involving objective functions z_i and z_j, $i, j = 1, \ldots, m$, $i \neq j$, is defined as

$$T_{ij}^G(\bar{z}) := \sup_{z \in Z_j^<(\bar{z})} \frac{z_i - \bar{z}_i}{\bar{z}_j - z_j}, \tag{2.8}$$

where $Z_j^<(\bar{z}) = \{z \in Z : z_j < \bar{z}_j, z_i \geq \bar{z}_i, i = 1, \ldots, m, i \neq j\}$. If $Z_j^<(\bar{z}) = \emptyset$, then $T_{ij}^G(\bar{z}) := \infty$ for all $i = 1, \ldots, m, i \neq j$.