INTRODUCTION TO CIRCLE PACKING

The Theory of Discrete Analytic Functions

KENNETH STEPHENSON
University of Tennessee

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INTRODUCTION TO CIRCLE PACKING

The topic of "circle packing" was born of the computer age but takes its inspiration and themes from core areas of classical mathematics. A circle packing is a configuration of circles having a specified pattern of tangencies, as introduced by William Thurston in 1985. This book lays out their study, from first definitions to the latest theory, computations, and applications. The topic can be enjoyed for the visual appeal of the packing images – over 200 in the book – and the elegance of circle geometry, for the clean line of theory, for the deep connections to classical topics, or for the emerging applications. Circle packing has an experimental and visual character that is unique in pure mathematics, and the book exploits that character to carry the reader from the very beginnings to links with complex analysis and Riemann surfaces. There are intriguing, often very accessible, open problems throughout the book and seven Appendices on subtopics of independent interest. This book lays the foundation for a topic with wide appeal and a bright future.

Preface

The circle is arguably the most studied object in all of mathematics, so it was a surprise at a conference in 1985 to hear William Thurston introducing a new topic called circle packing. And if encountering a new idea is one of the pleasures of mathematics, then seeing it attach to your favorite topic is a true joy. When Thurston conjectured a connection between circle packing and the venerable Riemann Mapping Theorem of 1851, I was hooked.

Now, nearly 20 years later, one sees that this was no mere passing encounter for these topics. Circle packing has opened a discrete world that both parallels and approximates the classical world of conformal geometry – a "quantum" complex analysis that is classical in the limit. In this book, I introduce circle packing as a portal into the beauties of conformal geometry, while I use the classical theory as a roadmap for developing circle packing.

Circle packings are configurations of circles with specified patterns of tangency. They should not be confused with sphere packings; here, it is the pattern of tangencies that is central – the connection between combinatorics and geometry. We will study the existence, uniqueness, computation, manipulation, display, and application of circle packings from the ground up. There are no formal prerequisites for this study; indeed, I shamelessly exploit the visual nature of circle packing and our native intuition about circles so that even the novice mathematician can penetrate deeply into the subject. At the same time, I have an obligation to circle packing itself as a new field, so the book is mathematically rigorous.

To balance access with rigor, I have structured the book in four parts. For most readers, Part I will be the first encounter with circle packings, so it is a broad overview: from a circle packing managerie to the function-theory paradigm. We become more formal in Part II with a proof of the fundamental existence and uniqueness result for maximal packings from first principles. Because the key roles are played by surprisingly elementary geometric arguments, this can serve even the non-mathematician as an exemplar of a robust and self-contained mathematical theory. In the classroom, Part II serves well as a one semester course for advanced undergraduate or beginning graduate students.

I define a discrete analytic function theory based on circle packings in Part III. At its core, analyticity is a profoundly geometric property, and this comes out in the discrete setting in a very compelling way. The amazing integrity of the analogies is confirmed in Part IV, when we prove that under refinement, the objects of the discrete theory converge to their classical counterparts. We prove Thurston's 1985 conjecture on the approximation of conformal maps (the Rodin/Sullivan Theorem) from first principles and then extend it broadly to other functions and to conformal structures. The circle packing methods described here have made the 150-year-old Riemann Mapping Theorem computable in many situations for the very first time. I demonstrate a number of applications, the most surprising of which involves "cortical brain mapping." Material in Parts III and IV could augment the traditional complex analysis sequence or serve as a basis for advanced topics courses.

The book also provides a wealth of material for individual or group projects from the undergraduate to the research level. I promote an intuitive and hands-on approach throughout, posing many open questions and experimental opportunities; see in particular, the several independent topics in the appendixes. I have provided "practica" on computational and software issues for those willing to get their hands dirty, and one can always download and run my software package, `CirclePack`. The book closes with a full circle packing bibliography.

People are drawn to mathematics for any number of reasons, from the clarity in elementary geometry, the challenge of richly layered theory and open questions, the discipline of computation, to the need for results in other areas. Read with an open mind and you can find all of these in circle packing – and I have not even mentioned the pure aesthetic pleasure of the pictures, which can sustain us all through the rough patches. I hope you enjoy circle packing.

I have many people to thank for their advice, encouragement, and patience over the years of this book's writing. Thanks go to my circle packing collaborators and friends, from whom I've learned so much: Dov Aharonov, Phil Bowers, Charles Collins, and Alan Beardon; also to my former students Tomasz Dubejko and Brock Williams, great circle packers both. Special thanks to Alan Beardon and Fred Gehring for their unfailing encouragement and sage advice over the years; to Oded Schramm, Jim Cannon, and Bill Floyd for many enjoyable and insightful conversations; and to William Thurston for the audacious notion of circle packing. To my many friends in classical function theory: I'm still one of you!

I began writing during a sabbatical at the University of Cambridge; my thanks to the department and, particularly, to Alan Beardon, Keith Carne, and my part-III class. Of course, this project could never have succeeded without the continued support and encouragement of wonderful colleagues and staff here at the University of Tennessee. Thanks to the circle packing class who helped me hone my notes: James Ashe, Matt Cathey, Denise Halverson, and Jason Howard. Finally, I acknowledge a debt of gratitude to the National Science Foundation for its financial support over the years and, likewise, to the Tennessee Science Alliance.

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I

Circle Packing

Circle packing has arrived so recently on the mathematical scene as to be totally new to many readers. Therefore, I am devoting Part I to an informal and largely visual tour of the topic, introducing only the most basic terminology and notation, but giving the reader a glimpse of how the full story will unfold. We all understand circles, but the reader may be surprised at how deeply they can carry us into the heart of conformal geometry.

Chapter 1 begins with a visit to a "menagerie" of circle packings, a wide-ranging collection that I hope you enjoy as much in the touring as I did in the collecting. The Menagerie suggests our first theme: Given a specified combinatoric pattern, what can one say about the existence, uniqueness, and variety of circle packings having those combinatorics? In Chapter 2 we get a view of the landscape beyond existence and uniqueness, where the central theme of the book – the emergence of fundamental parallels with analytic functions – plays out. I hope to set a style here that will carry on throughout the book, namely, one that engages the reader's native intuition not about static pictures, but about packing dynamics: How does a packing react if we change its boundary radii? If we introduce branching? How are the combinatorics and geometry feeding off one another? Deep classical themes will bubble to the surface with a surprising ease and clarity if one only remains open to the possibilities.

At the end of Part I is the first of four practica inviting the reader into the experimental side of the topic. Circle packings exist both in theory and in *fact*. The book requires nothing more than mental experiments, but those with an adventurous spirit may wish to grapple with my software package `CirclePack` or even do their own coding.

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1

A Circle Packing Menagerie

1.1. First Views

In the belief that images speak louder than words, I will begin with a preliminary ramble through a menagerie of circle packings. Look for the common features and the differences in preparation for the guided tour to follow.

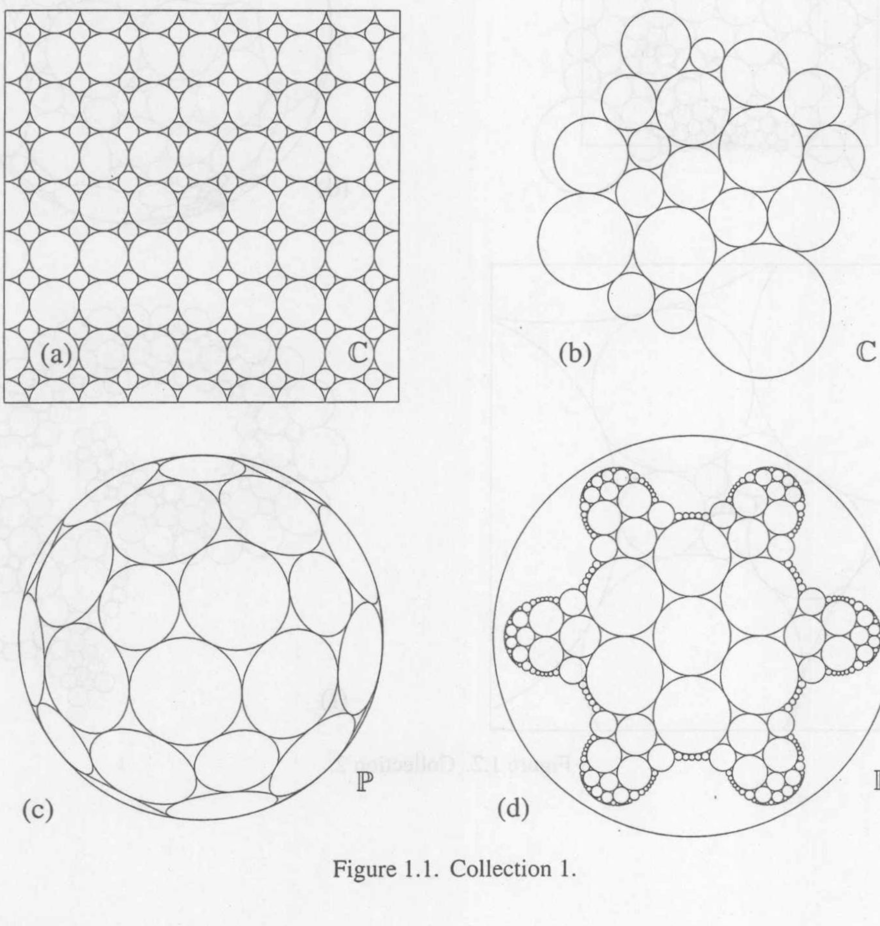

Figure 1.1. Collection 1.

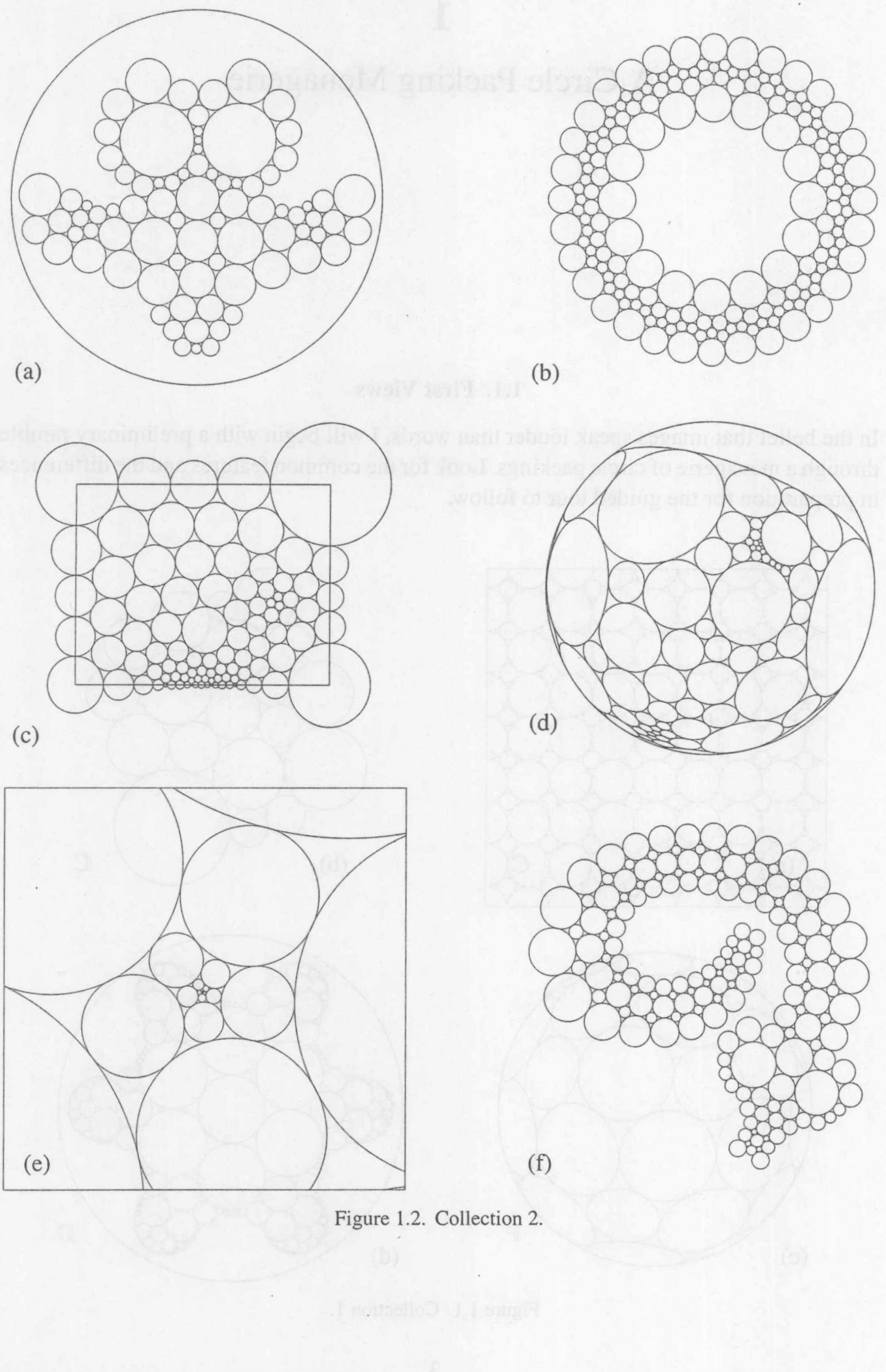

Figure 1.2. Collection 2.

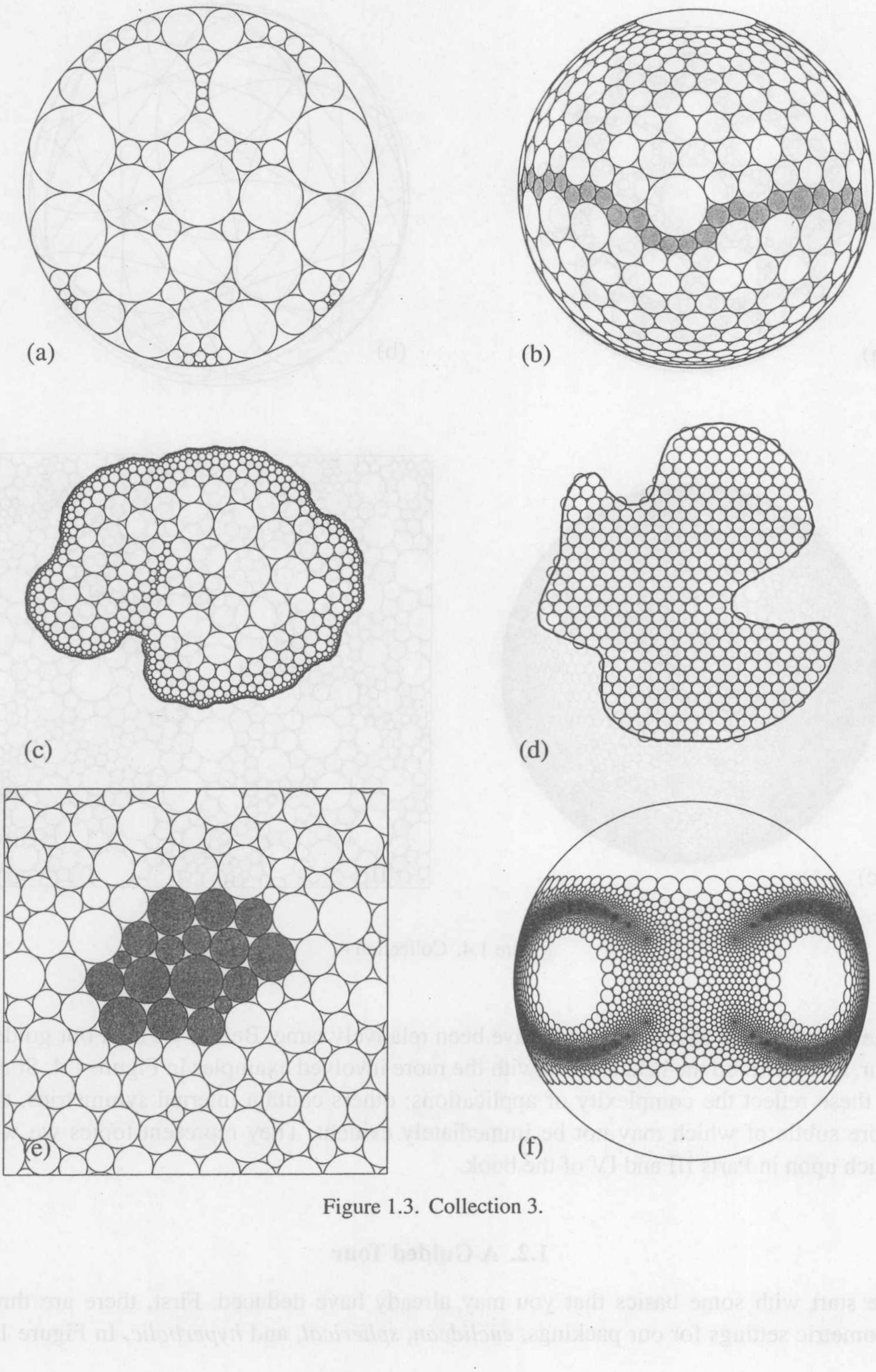

Figure 1.3. Collection 3.

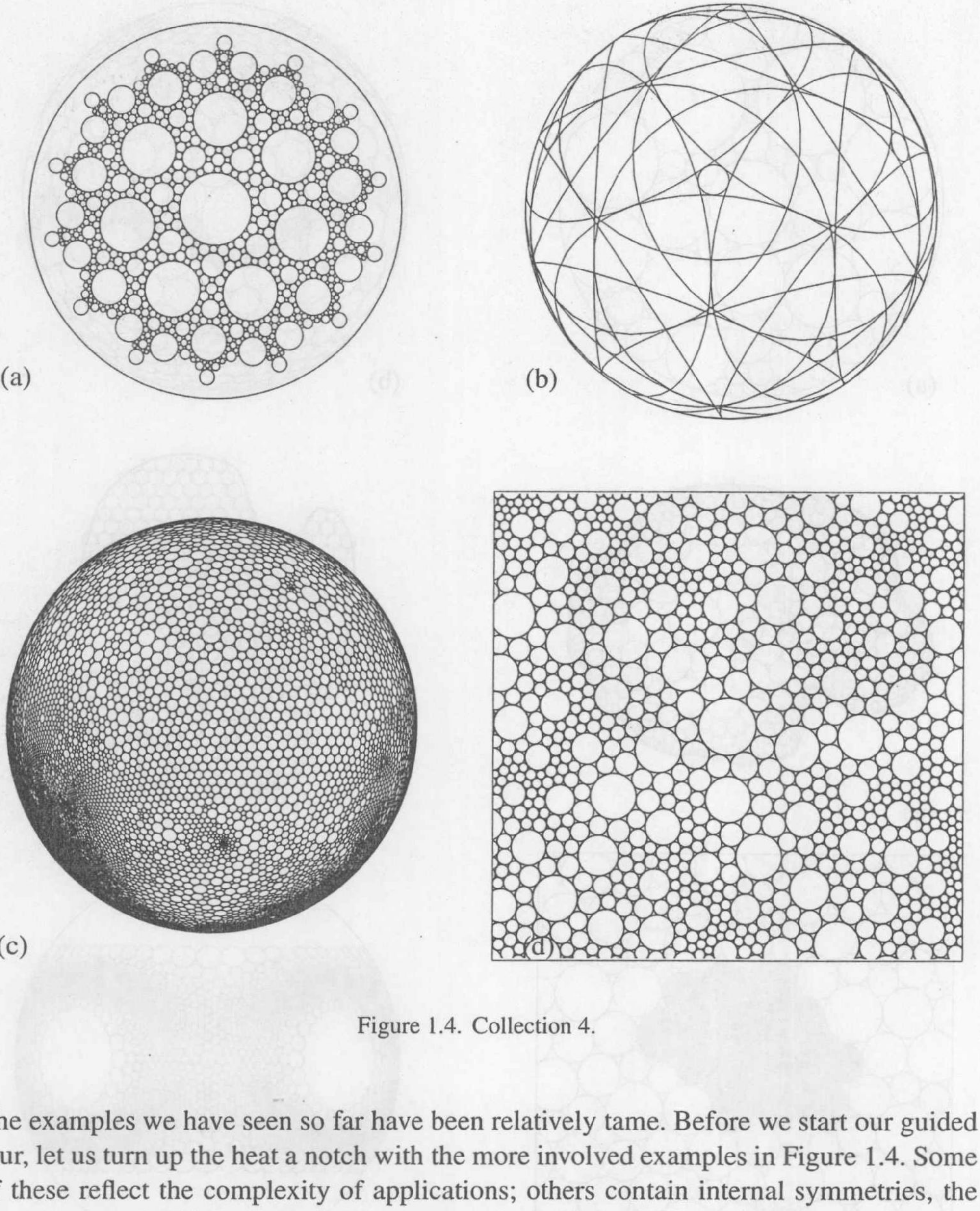

Figure 1.4. Collection 4.

The examples we have seen so far have been relatively tame. Before we start our guided tour, let us turn up the heat a notch with the more involved examples in Figure 1.4. Some of these reflect the complexity of applications; others contain internal symmetries, the more subtle of which may not be immediately evident. They represent topics we will touch upon in Parts III and IV of the book.

1.2. A Guided Tour

We start with some basics that you may already have deduced. First, there are three geometric settings for our packings, *euclidean*, *spherical*, and *hyperbolic*. In Figure 1.1

we see the familiar *euclidean plane* $\mathbb{R}^2$ in (a) and (b), the sphere in (c), and the interior of a disc in (d). Throughout the book, we treat $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and make use of complex arithmetic. The sphere will be the *Riemann sphere*, represented as the ordinary unit sphere in $\mathbb{R}^3$ and denoted by $\mathbb{P}$ (for *complex projective space*). The outer circle enclosing the packing of Figure 1.1(d) is not part of the packing; rather it is the boundary of the unit disc $\mathbb{D}$ in the plane. Here, however, $\mathbb{D}$ represents the Poincaré disc, a standard model of the *hyperbolic plane*. The geometries of $\mathbb{P}$, $\mathbb{C}$, and $\mathbb{D}$ will be of central importance in our work and we will have more to say about their distinct personalities shortly.

Next, observe that each packing involves an underlying pattern of tangencies. The hierarchy of structures is indicated in Fig. 1.5. All our tangencies are *external*, each circle lying outside the disc bounded by the other. In fact, however, tangencies do not occur in isolated pairs; rather the fundamental units of the patterns are mutually tangent triples of circles (*triples*, for short), with each triple forming a (curvilinear) triangular *interstice*. Triples are as important to the rigidity associated with circle packings as cross-bracing is to the rigidity of a bookcase. In turn, the triples of a pattern are linked together through shared pairs of circles to form the next level of structure, the *flower*, consisting of a central circle and some number of *petal* circles, the chain of successively tangent neighbors. The number of petals defines the *degree* of the central circle. The condition that every

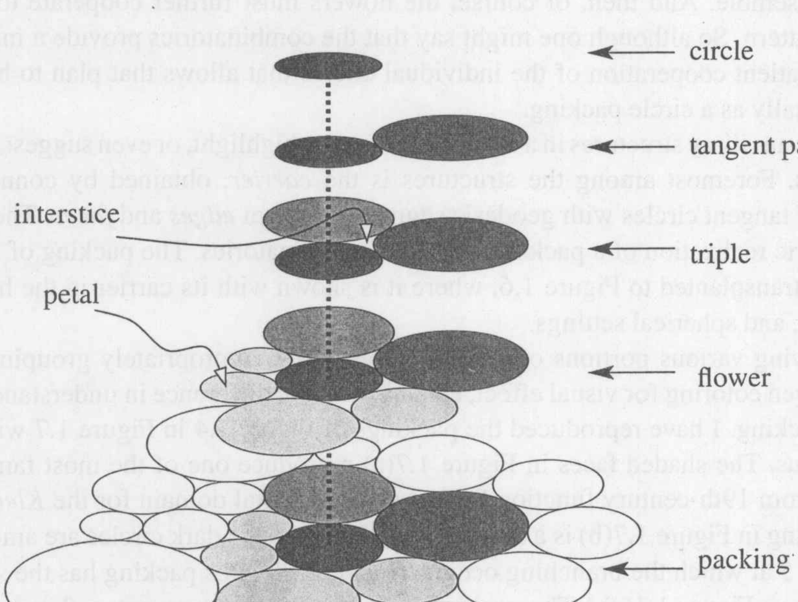

Figure 1.5. A hierarchy of circle packing structure.