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LINEAR RAY AND WAVE OPTICS IN PHASE SPACE

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Bridging Ray and Wave Optics via the Wigner Phase-Space Picture

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To my mother
whose caress still lingers on my cheek.
To my father
whose smile still shines into my eyes.
To my country
whose colours feed my avid love of life.

Preface

...
Amo i colori, tempi di un anelito
inquieto, irresolvibile, vitale
spiegazione umilissima e sovrana
dei cosmici "perché" del mio respiro.
...

A. Merini, *Colori*
(from A. Merini, *Fiore di poesia*, Einaudi, Torino, 1998)

Ray, wave and quantum concepts are central to diverse and seemingly incompatible models of light. Each model particularizes a specific "manifestation" of light, and then corresponds to adequate physical assumptions and formal approximations, whose domain of applicability is well established. Accordingly each model comprises its own set of geometrical and dynamical postulates with the pertinent mathematical means.

Geometrical optics models the light field as an aggregate of incoherent light rays, naïvely perceived as the trace of the motion of the "luminous corpuscles", which, emitted by the source, move through space in obedience to the usual laws of mechanics. It treats light rays as lines in 3-space dimensions and is accordingly concerned with the dynamical laws establishing how the rays bend when propagating in inhomogeneous media described by the refractive index function. Geometrical optics is not suited to explain interference, diffraction and quantum coherence effects, but, in contrast, it provides a particularly convenient means for the design of optical systems, which is based on the purely geometrical rules of ray tracing.

Geometrical optics has developed its own mathematical framework, which can remarkably be brought into correspondence with that of the Hamiltonian mechanics of point-particles, with "time" corresponding to the arc-length along the ray path and the mechanical "potential" to the refractive index of the optical medium.

Wave optics accounts for the wave characteristics of light. Originating directly from the classical electromagnetic theory, it shares with this theory the same system of theoretical principles and methods, which can notably be put in correspondence with those proper to relativistic quantum mechanics. Then, the geometry of light rays is replaced by the geometry of "luminous" waves, whose propagation is geometrically pictured as the transfer of the interference shaped vibrations from one portion of the medium to the contiguous one.

Wave optics treats the light waves as complex functions of position in 3-space dimensions and is accordingly concerned with the dynamical laws establishing how the wave function changes as the optical wave propagates through inhomogeneous media.

Quantum optics recovers the grainy view of geometrical optics, picturing the light ray as a stream of particle-like entities, the photons. Whereas geometrical optics deals with the influence at a macroscopic level of the medium on the trajectory of the photon streams, quantum optics is typically concerned with the wave-like question relevant to the coherence properties of the photon beams and to the relevance of those properties on the interaction of light with matter, which can correspondingly be treated quantum mechanically. Coherent and squeezed states of light are the building concepts of quantum optics.

Wigner optics bridges between ray and wave optics. It offers the optical phase space as the ambience and the Wigner function based technique as the mathematical machinery to accommodate between the two opposite extremes of light representation: the localized ray of geometrical optics and the unlocalized wave function of wave optics.

Notably quantum optics finds a convenient formulation in the proper phase space with the consequent geometrical view of coherent and squeezed states as circles and ellipses. The Wigner function methods can suitably be applied to quantum optics as well to enable effective analytical means for calculating expectation values and transition probabilities for the aforementioned states.

The purpose of the book is to introduce the reader to the optical phase-space and to the approaches to optics based on the Wigner distribution function, that have been developed over the past 25 years or so in several scientific titles. These yield the formal context, where concepts and methods of both ray and wave optics coalesce into a unifying formalism. In this respect, emphasis is given to the Lie algebra representation of optical systems and accordingly to the Lie algebra view of light propagation through optical systems.

The book is made as self-contained as possible. Chapter 1 presents the Hamiltonian equations of motion, which are basic to the development of both the transfer matrix formalism, appropriate to paraxial ray optics (Chapters 2 and 3), and the transfer operator formalism, suited to paraxial wave optics (Chapters 4 and 5). The relation of both formalisms to the Lie algebra methods is gently displayed.

Chapter 6 introduces the Wigner distribution function, elucidating its origin taken in quantum mechanics and illustrating its properties. A host of diverse optical signals are considered and the relevant Wigner distribution functions are analytically evaluated and graphically shown to help the intuitive perception of the simultaneous account of the signal in the space and spatial frequency domains, conveyed by the Wigner distribution function. Chapter 7 frames the Wigner distribution function within the broad realm of the phase-space signal representations, and illustrates the procedure, and the relevant optical architectures, for displaying the Wigner distribution function of a given signal. In Chapter 8 the laws for the transfer of the Wigner distribution function through linear optical systems are derived. Attention is drawn to the relation between the Wigner distribution function and the fractional Fourier transform, which is a revealing and effective tool for the space-frequency representation of signals (optical or not). Chapter 9 is concerned with the moments of the Wigner distribution function and their propagation laws.

The Wigner representation is presented on the fascinating border-line between quantum mechanics and signal theory.

Chapters are made as self-consistent as possible. Indeed, the Introduction to each chapter is conceived as a summary of the basic results of previous chapters, which are central to those that are going to be presented. A basic role is assigned to the diagrams, which illustrate the syllabus of each chapter, and the figures, which confer physical reality to conceptual architectures. A wide bibliography is given in relation to topics both carefully investigated and briefly mentioned.

Throughout the book the calculations are kept at an accessible level; most mathematical steps are justified. Difficulties might be encountered in connection with the algebra of operators, which do not obey the familiar rules of the algebra of scalars. Careful and illustrative comments on the peculiar behavior of operators are provided in § 1.4.1 in order to help the readers who are not acquainted with the operator algebra.

It is my hope to give the flavour of the fascinating feature of optics that enables a visible account of abstract mathematical entities, like, for instance, symplectic matrices and metaplectic operators, represented through integral transforms. Symplectic matrices and integral transforms, which essentially provide the formal structures for the considerations developed in Chapters 1 to 5, are intimately related, being indeed different representations of the same $Sp(2,\mathbb{R}) \sim Mp(2,\mathbb{R})$ group element. Firstly recognized within a purely quantum mechanical context, this relation has been applied in optics in connection with the fractional Fourier transform. The link between ray matrices and transfer operators from the alternative viewpoint of linear canonical transformations and relevant representations, is elucidated in § 5.6. This is an example of those *parallel paths*, that, explicitly illustrated or implicitly suggested in the text or in the problems, are intended to improve the feeling for the specific topic under consideration and to gain some insight and intuition for unforeseen correspondences and analogies between totally different physical problems.

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A joyful "Thank you, sorellina" is directed to Dr. F. Mucci, for enthusiastically listening to the description of my "conceptual castles". A bow is for the friends who share my passion for the theatre, for forgiving my absences from the prelims, being forgetfully enraptured in "mie adorate formuline".

It is with intense emotion that my thanks goes also to zia Aida and Maria, who could not see the completion of the book.

...
I love colours, times of a yearning
restless, irresolvable, vital,
very humble and supreme explanation
of the cosmic "why" of my breath.
...

Colours

Contents

1

Hamiltonian Picture of Light Optics. First-Order Ray Optics

1.1 Introduction

The phase space representation of light optics naturally arises from the Hamiltonian formulation of geometrical optics. Geometrical optics gives a simple model for light behaviour, in which the wave character of light is ignored. It is valid whenever light waves propagate through or around objects which are very large compared to the wavelength of the light and when we do not examine too closely what is happening in the proximity of shadows or foci. Accordingly, it does not account for diffraction, interference or polarization effects. Geometrical optics employs the concept of *light ray* [1], which we may give the naïve view as an infinitesimally thin beam of light. Several formal definitions of light ray have been elaborated within both the corpuscular and wave theory to accommodate geometrical abstraction and physical observability. All definitions work well in certain situations, but in others are confronted with intrinsically physical difficulties. Thus, for instance, the corpuscular view of rays as trajectories of "luminous" corpuscles confronts with the problem that the energy density may become infinite. Likewise the wave-like view of rays as orthogonal trajectories to the phase fronts of the light wave confronts with the difficulty of individualizing a defined wave front in the two-wave overlap distribution. Indeed, the ray must be thought of as a convenient and successful model which supports our perception, and hence facilitates the formal description, of a wide class of light phenomena. Geometrical optics establishes the geometrical rules governing the propagation of light rays through optical systems.

The analogy of geometrical optics of light rays to Hamiltonian mechanics of material particles is well established and effectively exploited. The *Hamiltonian formalism* was originally developed by Hamilton for optics in his 1828 paper *Theory of Systems of Rays* and in subsequent papers and brief notes, published during the years from 1830 to 1837 [2.1]. In his papers, Hamilton formulates the problem of studying the geometry of light rays as they pass through optical systems in terms of well-defined relations between the local coordinates of the rays entering and emerging from the system, specified with respect to the optical axis and properly chosen planes across the axis. He shows that, if the ray coordinates are suitably defined, the input-output relations configure as *symplectic* transformations, generated by a function of the ray variables, the *characteristic function*, whose functional form is determined solely by the optical properties of the system. Later, Hamilton realized that the same method could be applied unchanged to mechanical systems, replacing the optical axis by the time axis, the light rays by the particle trajectories and the ray-coordinates by the mechanical phase-space variables [2.2].

The phase space representation is a familiar method within the Hamiltonian formulation of classical mechanics, which describes the dynamics of a mechanical system with m degrees of freedom in terms of m generalized independent coordinates $(q_1, q_2, ..., q_m)$ and the same number of canonically conjugate variables $(p_1, p_2, ..., p_m)$ [3]. The mechanical phase space is the Cartesian space of these $2m$ coordinates. For example, the state of a free particle at a certain time is represented in the proper $6D$ phase space by a *representative point*, specified by the Cartesian coordinates $\mathbf{q} = (q_x, q_y, q_z)$ and the relevant momenta $\mathbf{p} = (p_x, p_y, p_z)$. The motion of the particle in real space corresponds to a trajectory in phase space. Then, the state of an ensemble of identical and noninteracting particles at a given time corresponds to a set of points in the $6D$ phase space. The domain occupied by this set of points moves through phase space as the particles move in real space. However, as the total number of particles remain constant, so will the total number of phase space points. Evidently a real density can be associated with the representative points in phase space, and correspondingly a distribution function of density $\rho(\mathbf{q}, \mathbf{p}, t)$ can be defined so that $\rho(\mathbf{q}, \mathbf{p}, t)dV$ specifies the number of representative points in the element of volume dV in the vicinity of the point $(\mathbf{q}, \mathbf{p})$. Liouville's theorem states the invariance of the density of representative points along the trajectory of any point, and accordingly of the volume of the phase space domain, even though its shape may change considerably during the motion.

Likewise the geometric-optical phase space is the $4D$ Cartesian space of the ray position and momentum coordinates (q_x, q_y, p_x, p_y). However, the phase spaces of classical mechanics and geometrical optics are globally different. The particle momentum of classical mechanics is not restricted in value, whilst the ray momentum of geometrical optics is confined within a circle determined by the local refractive index through the inherent form of the optical Hamiltonian. In the linear approximation the ray momentum is assumed to range well below its natural limit, which then is ignored. Thus the geometric-optical phase space of linear optics comes to be similar to the mechanical phase space.

Section 1.2 reviews the Hamiltonian formulation of geometrical optics and

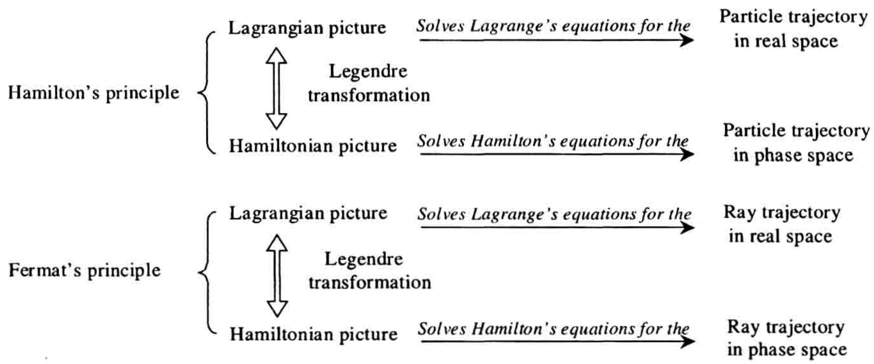

FIGURE 1.1. The Fermat extremal principle based formulation of geometrical optics mirrors that of classical mechanics, based on the Hamilton minimal principle.

introduces the related concept of geometric-optical phase space. Section 1.3 emphasizes the symplectic nature of ray propagation, and details the suited mathematical settings (Poisson brackets and Lie operators) to approach the integration of Hamilton's equations for the light ray. In Sect. 1.4 the ray-transfer operator is introduced and the relevant Lie-transformation based formalism is described. Illustrative examples of phase-space transformations are given in Sect. 1.5. Sections 1.6 and 1.7 illustrate the linear approximation to light-ray propagation, which naturally yields the ray-transfer matrix formalism. Finally, Sect. 1.8 clarifies the link between the ray-matrix approach and the cardinal point (and planes) method.

1.2 Hamiltonian picture of light-ray propagation

We will give a brief account of the Hamiltonian formulation of geometrical optics in order to fix the notations we adopt and to trace the conceptual path towards the phase space representation and the inherent geometry.

Hamiltonian optics develops from Fermat's principle of extremal optical path, which is the optical analog of Hamilton's principle of least action (Fig. 1.1). From Hamilton's principle one can derive both the Lagrangian and Hamiltonian mechanics, related through the Legendre transformation [3]. Likewise from Fermat's principle one can develop the Lagrangian as well as the Hamiltonian formulation of optics [4]. The former yields the equations for the ray variables in real space, while the latter the equations for the ray variables in phase space. We will cursorily illustrate the basic steps leading to both

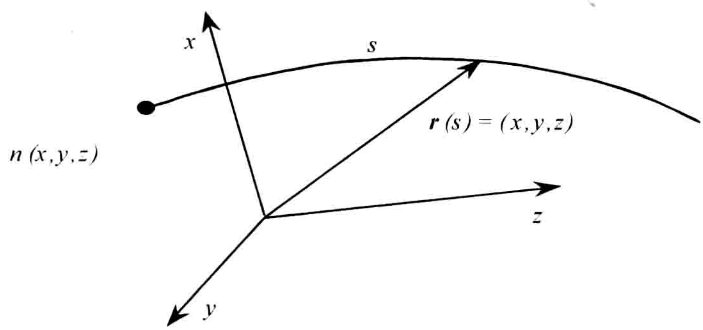

FIGURE 1.2. Geometrical optics describes the medium by the refractive index function $n(x, y, z)$ and the light rays by the 3-vector of functions $\mathbf{r}(s) \equiv (x(s), y(s), z(s))$ of the arc length s measured along the ray path.

pictures, addressing the reader to [4] for a more detailed treatment.

As a natural scenario for introducing Fermat's principle [4, 5], we consider an inhomogeneous medium, occupying a certain region in the $3D$ space, where we suppose a Cartesian system of coordinates (x, y, z) be assigned. The optical properties of the medium are typically described by the refractive index $n(x, y, z)$, given as a scalar function of space[a]. A light ray is propagating in the medium along some trajectory. Regarded as a line in the $3D$ space, the ray can accordingly be described by the position vector $\mathbf{r}(s) \equiv (x(s), y(s), z(s))$ for points on the ray, with the coordinates being functions of the arc length s measured along the ray path with respect to a chosen point (Fig. 1.2).

Fermat's principle combines the geometrical and physical aspects of the ray propagation through the concept of *optical path.*

We recall that given two points P_1 and P_2 and a curve C connecting them, the *geometrical* path length $\mathfrak{L}(C)$ from P_1 to P_2 along C is defined as the length of C and hence is formally given by the line integral

$$\mathfrak{L}(C) = \int_{P_1}^{P_2} ds, \qquad (1.2.1)$$

performed along C from P_1 to P_2; s denotes the arc length measured along the path and $ds = \sqrt{dx^2 + dy^2 + dz^2}$ is the infinitesimal arc length.

Correspondingly, the *optical* path length $\mathcal{L}(C)$ along the ray trajectory C

[a] We will consider only linear spatially nondispersive isotropic media, whose refractive index function is accordingly dependent on position and independent of direction. Hence we will distinguish only between homogeneous and inhomogeneous media, according to whether the scalar index function is uniform or changes from point to point within the medium.