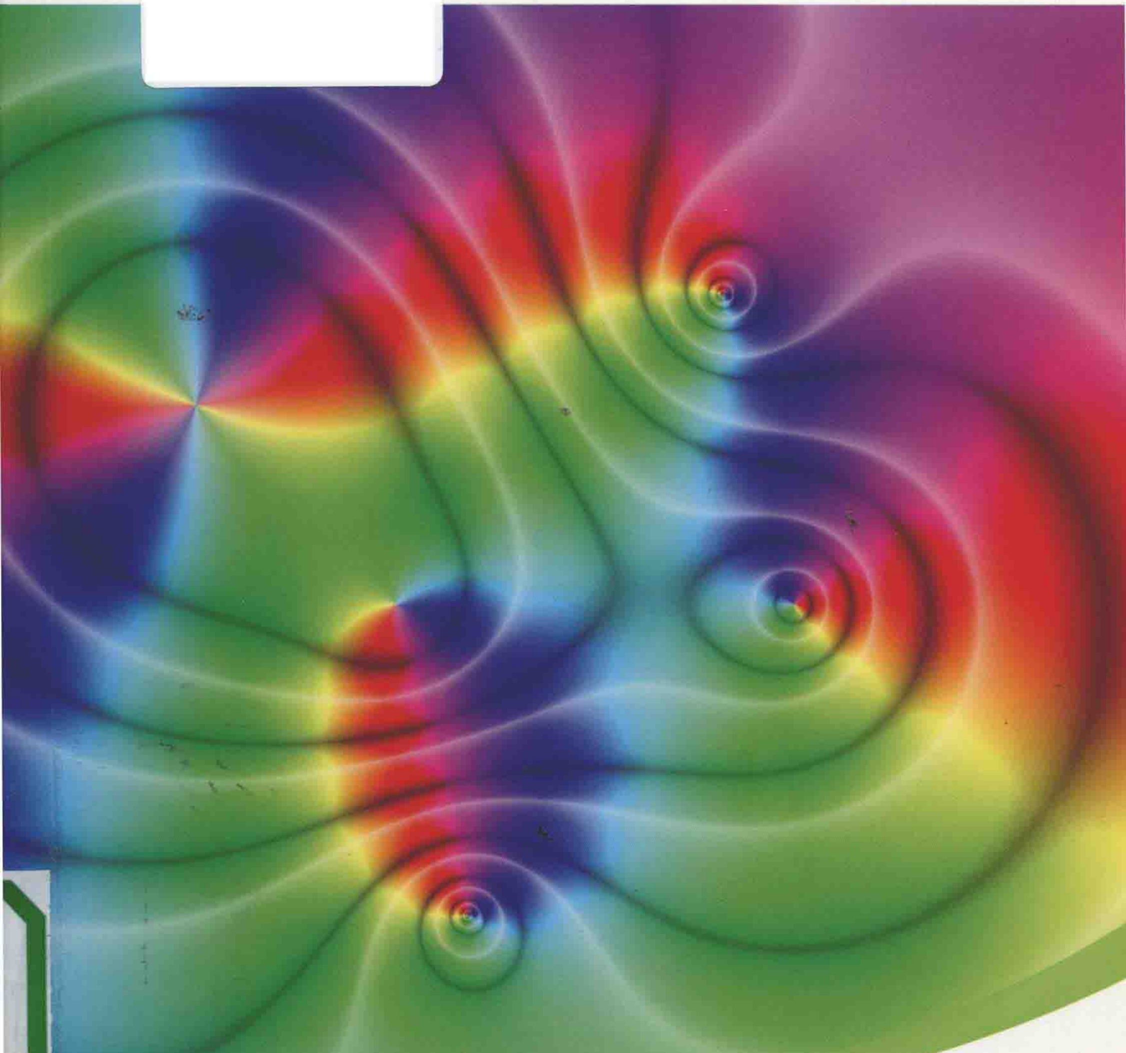
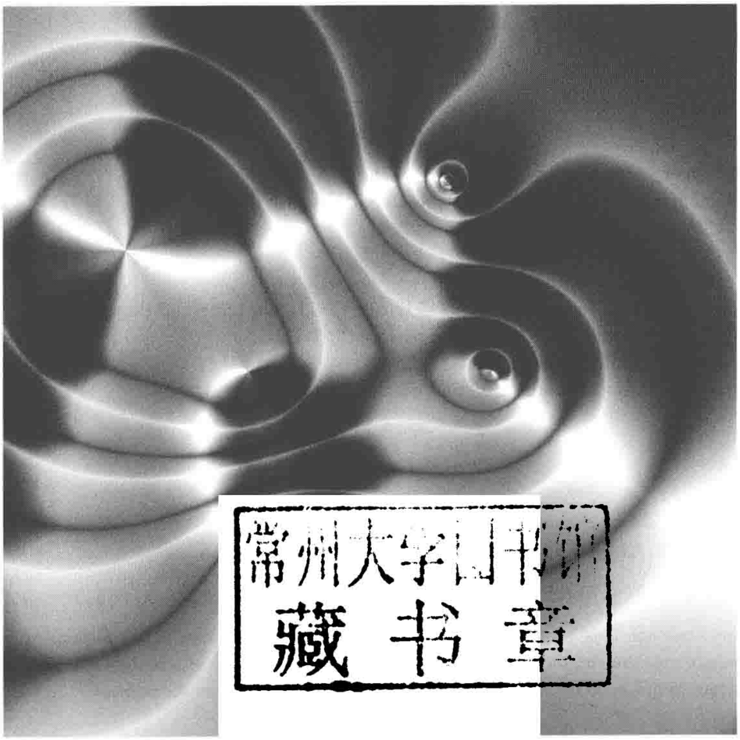
A FRIENDLY APPROACH TO COMPLEX ANALYSIS

Sara Maad Sasane • Amol Sasane



World Scientific

A FRIENDLY APPROACH TO COMPLEX ANALYSIS



Sara Maad Sasane
Lund University, Sweden

Amol Sasane
London School of Economics, UK

World Scientific

NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Credit for Image from Wikimedia Commons:
Color plot of complex function $(x^{2-1}) * (x-2-I)^2 / (x^2+2+2I)$, hue represents the argument, sat and value represents the modulus (Permission=CC-BY 2.5).
Image creator: Claudio Rocchini
Source: http://en.wikipedia.org/wiki/File:Color_complex_plot.jpg

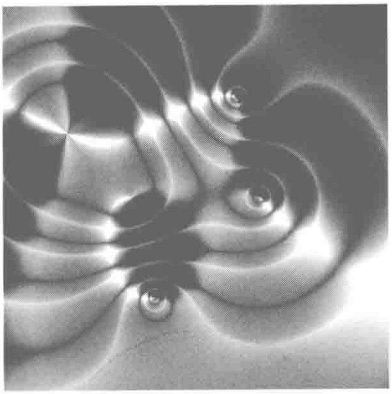
A FRIENDLY APPROACH TO COMPLEX ANALYSIS

ISBN 978-981-4578-98-1
ISBN 978-981-4578-99-8 (pbk)

Printed in Singapore by World Scientific Printers.

A FRIENDLY APPROACH TO

COMPLEX ANALYSIS



Preface

We give an overview of what complex analysis is about and why it is important. As the student must have learnt the notion of a complex number at some point, we will use *that* familiarity in our discussion here. Later on, starting from Chapter 1 onwards, we will start things from scratch again. So the reader should not worry about being lost in this preface!

What is Complex Analysis?

In *real* analysis, one studies (rigorously) calculus in the setting of real numbers. Thus one studies concepts such as the convergence of real sequences, continuity of real-valued functions, differentiation and integration. Based on this, one might guess that in *complex* analysis, one studies similar concepts in the setting of complex numbers. This is partly true, but it turns out that up to the point of studying differentiation, there are no new features in complex analysis as compared to the real analysis counterparts. But the subject of complex analysis departs radically from real analysis when one studies differentiation. Thus, complex analysis is not merely about doing analysis in the setting of complex numbers, but rather, much more specialized:

Complex analysis is the study of "complex differentiable" functions.

Recall that in real analysis, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at* $x_0 \in \mathbb{R}$ if there exists a real number L such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L,$$

that is, for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$, there holds that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \epsilon.$$

In other words, given any distance ϵ, we can make the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

lie within a distance of ϵ from the real number L for all x sufficiently close to, but distinct from, x_0.

In the same way, we say that a function $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable at* $z_0 \in \mathbb{C}$ if there exists a complex number L such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L,$$

that is, for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, there holds that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| < \epsilon.$$

The only change from the previous definition is that now the distances are measured with the *complex* absolute value, and so this is a straightforward looking generalization.

But we will see that this innocent looking generalization is actually quite deep, and the class of complex differentiable functions looks radically different from real differentiable functions. Here is an instance of this.

Example 0.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$

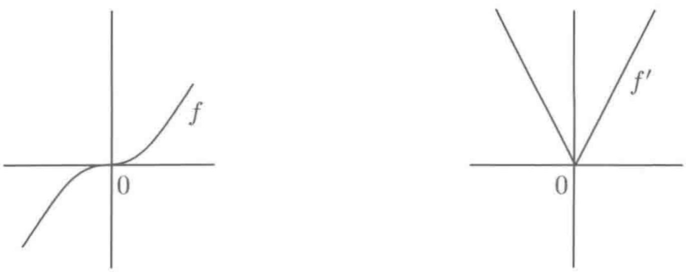


Fig. 0.1 Graphs of the functions f and its derivative f'.

Then f is differentiable everywhere, and

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ -2x & \text{if } x < 0. \end{cases} \tag{0.1}$$

Indeed, the above expressions for $f'(x)$ are immediate when $x \neq 0$, and $f'(0) = 0$ can be seen as follows. For $x \neq 0$,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x)}{x} \right| = \frac{|x|^2}{|x|} = |x| = |x - 0|,$$

and so given $\epsilon > 0$, we can take $\delta = \epsilon$ (> 0) and then we have that whenever $0 < |x - 0| < \delta$,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = |x - 0| < \delta = \epsilon.$$

However, it can be shown that f' is not differentiable at 0; see Exercise 0.1. This is visually obvious since f' has a corner at $x = 0$.

Summarizing, we gave an example of an $f : \mathbb{R} \to \mathbb{R}$, which is differentiable everywhere in $\mathbb{R}$, but whose derivative f' is not differentiable on $\mathbb{R}$.

In contrast, we will later learn that if $F : \mathbb{C} \to \mathbb{C}$ is a complex differentiable function in $\mathbb{C}$, then it is infinitely many times complex differentiable! In particular, its complex derivative F' is also complex differentiable in $\mathbb{C}$. Clearly this is an unexpected result if all we are used to is real analysis. We will later learn that the reason this miracle takes place in complex analysis is that complex differentiability imposes some "rigidity" on the function which enables this phenomenon to occur. We will also see that this rigidity is a consequence of the special geometric meaning of multiplication of complex numbers. ◊

Exercise 0.1. Prove that $f' : \mathbb{R} \to \mathbb{R}$ given by (0.1) is not differentiable at 0.

Why study complex analysis?

Although it might seem that complex analysis is just an exotic generalization of real analysis, this is not so. Complex analysis is fundamental in all of mathematics. In fact real analysis is actually inseparable with complex analysis, as we shall see, and complex analysis plays an important role in the applied sciences as well. Here is a list of a few reasons to study complex analysis:

(1) **PDEs**. If $f : \mathbb{C} \to \mathbb{C}$ is a complex differentiable function in $\mathbb{C}$, then we have two associated real-valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, namely the real and imaginary parts of f: for $(x, y) \in \mathbb{R}^2$, $u(x, y) := \text{Re}(f(x, y))$ and $v(x, y) := \text{Im}(f(x, y))$.

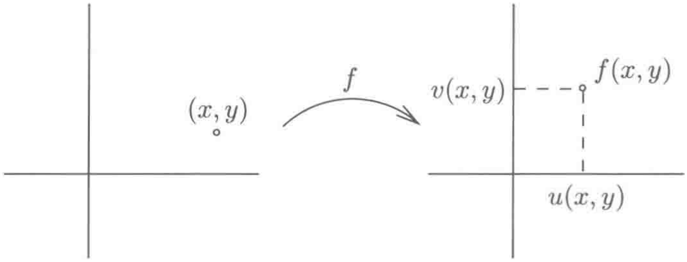


Fig. 0.2 The real and imaginary parts u, v of f.

It turns out that real and imaginary parts u, v satisfy an important basic PDE, called the Laplace equation:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly $\Delta v = 0$ in $\mathbb{R}^2$ as well. The Laplace equation itself is important because many problems in applications, for example, in physics, give rise to this equation. It occurs for instance in electrostatics, steady-state heat conduction, incompressible fluid flow, Brownian motion, etc.

(2) **Real analysis**. Using complex analysis, we can calculate some integrals in real analysis, for example

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx \quad \text{or} \quad \int_0^{\infty} \cos(x^2) dx.$$

Note that the problem is set in the reals, but one can solve it using complex analysis.

Moreover, sometimes complex analysis helps to clarify some matters in real analysis. Here is an example of this. Consider

$$f(x) := \frac{1}{1 - x^2}, \quad x \in \mathbb{R} \setminus \{-1, 1\}.$$

Then f has a "singularity" at $x = \pm 1$, by which we mean that it is not defined there. It is, however defined in particular in the interval $(-1, 1)$. The geometric series

$$1 + x^2 + x^4 + x^6 + \dots$$

converges for $|x^2| < 1$, or equivalently for $|x| < 1$, and we have

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1 - x^2} = f(x) \text{ for } x \in (-1, 1).$$

From the formula for f, it is not a surprise that the power series representation of the function f is valid only for $x \in (-1, 1)$, since f itself has singularities at $x = 1$ and at $x = -1$. But now let us consider the new function g given by

$$g(x) := \frac{1}{1 + x^2}, \quad x \in \mathbb{R}.$$

The geometric series $1 - x^2 + x^4 - x^6 + - \dots$ converges for $|-x^2| < 1$, or equivalently for $|x| < 1$, and we have

$$1 - x^2 + x^4 - x^6 + - \cdots = \frac{1}{1 + x^2} = g(x) \text{ for } x \in (-1, 1).$$

So the power series representation of the function g is again valid only for $x \in (-1, 1)$, despite there being no obvious reason from the formula for g for the series to break down at the points $x = -1$ and $x = +1$. The mystery will be resolved later on in this book, and we need to look at the *complex* functions

$$F(z) = \frac{1}{1 - z^2} \text{ and } G(z) = \frac{1}{1 + z^2}$$

(whose restriction to $\mathbb{R}$ are the functions f and g, respectively). In particular, G now has singularities at $z = \pm i$, and we will see that what matters for the power series expansion to be valid is the biggest size of the disk we can consider with center at $z = 0$ which does not contain any singularity of G.

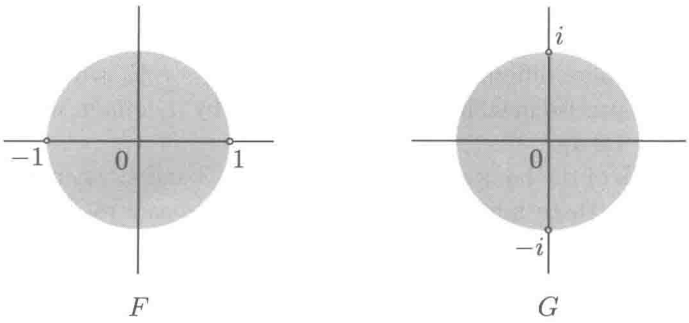


Fig. 0.3 Singularities of F and G.

(3) **Applications**. Many tools used for solving problems in applications, such as the Fourier/Laplace/z-transform, rely on complex function theory. These tools in turn are useful for example to solve differential equations which arise from applications. Complex analysis plays an important in applied subjects such as mathematical physics and engineering, for example in control theory, signal processing and so on.
(4) **Analytic number theory**. Perhaps surprisingly, many questions about the natural numbers can be answered using complex analytic tools. For example, consider the Prime Number Theorem, which gives an asymptotic estimate on the number $\pi(n)$ of primes less than n for large n:

Theorem 0.1. (Prime Number Theorem) $\displaystyle\lim_{n\to\infty}\frac{\pi(n)}{n/(\log n)}=1.$

It turns out that one can give a proof of the Prime Number Theorem using complex analytic computations with a certain complex differentiable function called the Riemann zeta function. Associated with the Riemann zeta function is also a famous unsolved problem in analystic number theory, namely the Riemann Hypothesis, saying that all the "nontrivial" zeros of the Riemann zeta function lie on the line $\text{Re}(s)=\frac{1}{2}$ in the complex plane. We will meet the Riemann zeta function in Exercise 4.5 later on.

What will we learn in Complex Analysis

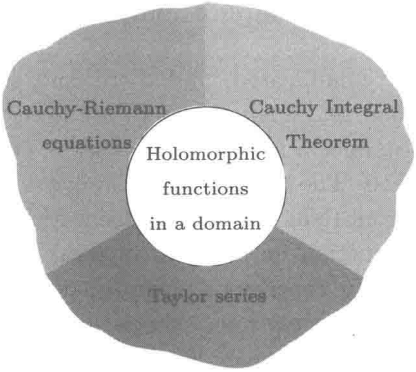
The central object of study in this course will be

holomorphic functions in a domain

that is, complex differentiable functions $f : D \to \mathbb{C}$, where D is a "domain" (the precise meaning of what we mean by a domain will be given in Subsection 1.3.4).

The bulk of the book is then in Chapters 2, 3 and 4, where we construct the following three lanterns to shed light on our central object of study, namely holomorphic functions in a domain:

(1) The Cauchy-Riemann equations,
(2) The Cauchy Integral Theorem,
(3) Taylor series.



The core content of the book can be summarized in the following Main Theorem[1]:

Theorem 0.2. *Let D be an open path connected set and let $f : D \to \mathbb{C}$. Then the following are equivalent:*

(1) *For all $z \in D$, $f'(z)$ exists.*
(2) *For all $z \in D$ and all $n \geq 0$, $f^{(n)}(z)$ exists.*
(3) $u := \text{Re}(f)$, $v := \text{Im}(f)$ *are continuously differentiable and* $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ *in D.*
(4) *For each simply connected subdomain S of D, there exists a holomorphic $F : S \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in S$.*
(5) *f is continuous on D and for all piecewise smooth closed paths γ in each simply connected subdomain of D, we have*

$$\int_\gamma f(z)dz = 0.$$

(6) *If $\{z \in \mathbb{C} : |z - z_0| \leq r\} \subset D$, then there is a unique sequence $(c_n)_{n\geq 0}$ in $\mathbb{C}$ such that for all z with $|z - z_0| < r$,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Furthermore, $c_n = \dfrac{1}{2\pi i} \displaystyle\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ *and* $c_n = \dfrac{f^{(n)}(z_0)}{n!}$.

[1]Don't worry about the unfamiliar terms/notation here: that is what we will learn, besides the proof!

Complex Analysis is not complex analysis!

Indeed, it is not very complicated, and there isn't much analysis. The analysis is "softer" than real analysis: there are fewer deltas and epsilons and difficult estimates, once a few key properties of complex differentiable functions are established. The Main Theorem above tells us that the subject is radically different from Real Analysis. Indeed, we have seen that a real-valued differentiable function on an open interval (a, b) need not have a continuous derivative. In contrast, a complex differentiable function on an open subset of $\mathbb{C}$ is infinitely many times differentiable! This happens because the special geometric meaning of complex multiplication implies that complex differentiable functions behave in a rather controlled manner locally infinitesimally, and aren't allowed to map points willy nilly. This controlled behaviour makes these functions rigid and we will see this in Section 2.3. Nevertheless there are enough of them to make the subject nontrivial and interesting!

The intended audience

These notes constitute a basic course in Complex Analysis, for students who have studied calculus in one and in several variables. The title of the book is meant to indicate that we aim to cover the bare bones of the subject with minimal prerequisites. The notes originated as lecture notes when the second author gave this course for third year students of the BSc programme in Mathematics and/with Economics.

Acknowledgements

Thanks are due to Raymond Mortini, Adam Ostaszewski and Rudolf Rupp for many useful comments. This book relies heavily on some of the sources mentioned in the bibliography. This applies also to the exercises. At some instances we have given detailed references in the section on notes at the end of each chapter, but no claim to originality is made in case there is a missing reference.

Sara Maad Sasane and Amol Sasane,
London and Lund, 2013

Contents

Preface v

1. Complex numbers and their geometry 1

1.1 The field of complex numbers 1
1.2 Geometric representation of complex numbers 5
1.3 Topology of $\mathbb{C}$ 11
1.3.1 Metric on $\mathbb{C}$ 12
1.3.2 Open discs, open sets, closed sets and compact sets 13
1.3.3 Convergence and continuity 14
1.3.4 Domains 15
1.4 The exponential function and kith 17
1.4.1 The exponential $\exp z$ 18
1.4.2 Trigonometric functions 22
1.4.3 Logarithm function 23
1.5 Notes 27

2. Complex differentiability 29

2.1 Complex differentiability 30
2.2 Cauchy-Riemann equations 36
2.3 Geometric meaning of the complex derivative 49
2.4 The d-bar operator 56
2.5 Notes 58

3. Cauchy Integral Theorem and consequences 59

3.1 Definition of the contour integral 59

3.1.1 An important integral 67
3.2 Properties of contour integration 69
3.3 Fundamental Theorem of Contour Integration 73
3.4 The Cauchy Integral Theorem 77
3.4.1 Special case: simply connected domains 82
3.4.2 What happens with nonholomorphic functions? . 85
3.5 Existence of a primitive 88
3.6 The Cauchy Integral Formula 91
3.7 Holomorphic functions are infinitely differentiable 96
3.8 Liouville's Theorem; Fundamental Theorem of Algebra . . 99
3.9 Morera's Theorem: converse to Cauchy's Integral Theorem 101
3.10 Notes 103

4. Taylor and Laurent series 105

4.1 Series 106
4.2 Power series 107
4.2.1 Power series and their region of convergence . . . 107
4.2.2 Power series are holomorphic 113
4.3 Taylor series 117
4.4 Classification of zeros 122
4.5 The Identity Theorem 126
4.6 The Maximum Modulus Theorem 129
4.7 Laurent series 131
4.8 Classification of singularities 141
4.8.1 Wild behaviour near essential singularities 151
4.9 Residue Theorem 153
4.10 Notes 162

5. Harmonic functions 163

5.1 What is a harmonic function? 163
5.2 What is the link between harmonic functions and holomorphic functions? 165
5.3 Consequences of the two way traffic: holomorphic $\leftrightarrow$ harmonic 170
5.3.1 Harmonic functions are smooth 171
5.3.2 Mean value property 172
5.3.3 Maximum Principle 172

5.4 Uniqueness of solution for the Dirichlet problem 173
5.5 Notes 176

Solutions 177

Bibliography 269

Index 271