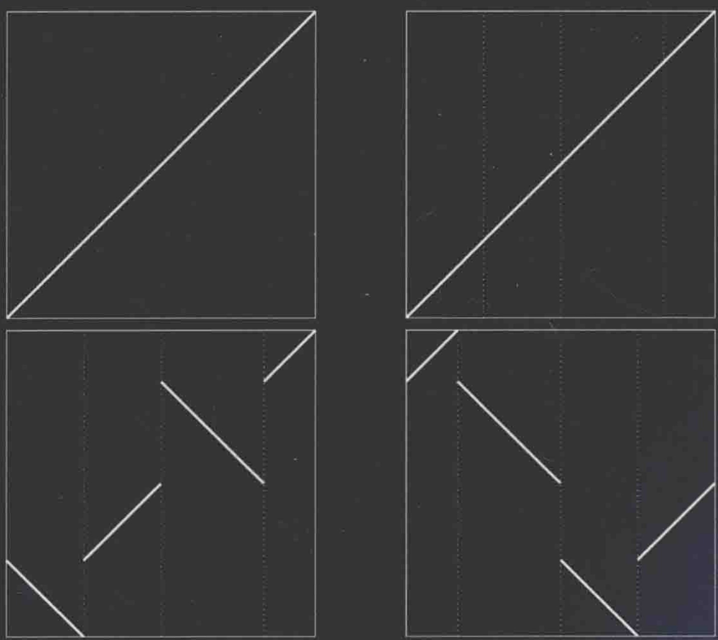
PRINCIPLES OF COPULA THEORY



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PRINCIPLES OF COPULA THEORY

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PRINCIPLES OF COPULA THEORY

Preface

The official history of copulas begins in 1959 with Sklar [1959]; but, as is often the case in Mathematics, for groundbreaking results there are forerunners and precedents. These latter ones involve asking the right question, proving a partial result, possibly in a special case and so forth. In the case of copulas, Hoeffding [1940, 1941] introduced a concept very close to that of a copula. He constructed what we now call a copula on the square $[-1/2, 1/2] \times [-1/2, 1/2]$ rather than on the unit square and gave some of the properties, including the bounds. Perhaps his papers did not initially attract the attention they deserved because they were published in a journal without a wide circulation during the war (see [Fisher and Sen, 1994]).

Slightly later Fréchet [1951] studied the following problem: given the probability distribution functions (d.f.'s) $F_1, \dots, F_d$ of d random variables $X_1, \dots, X_d$ defined on the same probability space, what can be said about the set $\Gamma(F_1, \dots, F_d)$ of d–dimensional d.f.'s whose marginals are $F_1, \dots, F_d$? The problem is hence related to the determination of a set of multivariate d.f.'s $\Gamma(F_1, \dots, F_d)$, now called the *Fréchet class*, given the partial information about the one-dimensional d.f.'s of $F_1, \dots, F_d$. In no case is the Fréchet class of $F_1, \dots, F_d$ empty, since one may always construct a multivariate d.f. in $\Gamma(F_1, \dots, F_d)$ assuming the independence among $F_1, \dots, F_d$, but, at the time, it was not clear which the other elements of $\Gamma(F_1, \dots, F_d)$ were.

For Fréchet's work see also [Dall'Aglio, 1972].

Preliminary studies about this problem were conducted by Féron [1956], Fréchet [1956], Gumbel [1958], but see also [Dall'Aglio, 1991] and [Schweizer, 1991] for a historical overview. In 1959, Sklar obtained the most significant result in this respect, by introducing the notion and the name of a *copula*[1] and proving the theorem that now bears his name [Sklar, 1959]. In his own words [Sklar, 1996]:

> [...] In the meantime, Bert (Schweizer) and I had been making progress in our work on statistical metric spaces, to the extent that Menger suggested it would be worthwhile for us to communicate our results to Fréchet. We did: Fréchet was interested, and asked us to write an announcement for the Comptes Rendus [Schweizer and Sklar, 1958]. This began an exchange of letters with Fréchet, in the course of which he sent me several packets of reprints, mainly dealing with the work he and his colleagues were doing on distributions with given marginals. These reprints, among the later arrivals of which I particularly single out that of Dall'Aglio [1959], were important for much of

[1] As for the grammatical meaning of the word, the electronic version of the Oxford English Dictionary gives: "*Logic and Gram.* That part of a proposition which connects the subject and predicate; the present tense of the verb to be (with or without a negative) employed as a mere sign of predication".

> our subsequent work. At the time, though, the most significant reprint for me was that of Féron [1956].
>
> Féron, in studying three-dimensional distributions, had introduced auxiliary functions, defined on the unit cube, that connected such distributions with their one-dimensional margins. I saw that similar functions could be defined on the unit n-cube for all $n \geq 2$ and would similarly serve to link n-dimensional distributions to their one-dimensional margins. Having worked out the basic properties of these functions, I wrote about them to Fréchet, in English. He asked me to write a note about them in French. While writing this, I decided I needed a name for these functions. Knowing the word "copula" as a grammatical term for a word or expression that links a subject and predicate, I felt that this would make an appropriate name for a function that links a multidimensional distribution to its one-dimensional margins, and used it as such. Fréchet received my note, corrected one mathematical statement, made some minor corrections to my French, and had the note published by the Statistical Institute of the University of Paris as [Sklar, 1959].[2]

The proof of Sklar's theorem was not given in [Sklar, 1959], but a sketch of it was provided by Sklar [1973] and, finally, showed in detail by Schweizer and Sklar [1974]. It turned out that for a few years practitioners in the field had to reconstruct it relying on the handwritten notes by Sklar himself; this was the case, for instance, of the second author. It should also be mentioned that some "indirect" proofs of Sklar's theorem (without mentioning copulas) were later discovered by Moore and Spruill [1975] and by Deheuvels [1978].

For about 15 years, most results concerning copulas were obtained in the framework of the theory of Probabilistic metric spaces [Schweizer and Sklar, 1983]. The event that arose the interest of the statistical community in copulas occurred in the mid-seventies, when Schweizer, in his own words [Schweizer, 2007],

> quite by accident, reread a paper by A. Rényi, entitled *On measures of dependence* and realized that [he] could easily construct such measures by using copulas.

See [Rényi, 1959] for Rényi's paper. These results were presented to the statistical community in the paper by Schweizer and Wolff [1981] (see also [Schweizer and Wolff, 1976; Wolff, 1977]).

However, for several years, Chapter 6 of the fundamental book by Schweizer and Sklar [1983], devoted to the theory of Probabilistic metric spaces and published in 1983, was the main source of basic information on copulas. Again in Schweizer's words from (Schweizer [2007]),

> After the publication of these articles and of the book ... the pace quickened as more ... students and colleagues became involved. Moreover, since interest in questions of statistical dependence was increasing, others came to the subject from different directions. In 1986 the enticingly entitled article *The joy of copulas* by Genest and MacKay [1986b], attracted more attention.

In 1990, Dall'Aglio organised the first conference devoted to copulas, aptly called "Probability distributions with given marginals" [Dall'Aglio et al., 1991]. This turned out to be the first in a series of conferences that greatly helped the development of the

[2] Curiously, it should be noted that in that paper, the author "Abe Sklar" is named as "M. Sklar".

field, since each of them offered the chance of presenting one's results and to learn those of other researchers; here we mention the conferences held in Seattle in 1993 [Rüschendorf et al., 1996], in Prague in 1996 [Beneš and Štěpán, 1997], in Barcelona in 2000 [Cuadras et al., 2002], in Québec in 2004 [Genest, 2005a,b]), in Tartu in 2007 [Kollo, 2009] and in Maresias, Brazil (2010). The Québec conference had a much larger attendance than the previous ones as a consequence of the fairly recent interest in copulas from the part of the investigators in finance and risk management.

In fact, at end of the nineties, the notion of copulas became increasingly popular. Two books about copulas appeared that were to become the standard references for the following decade. In 1997 Joe published his book on multivariate models [Joe, 1997], with a great part devoted to copulas and families of copulas. In 1999 Nelsen published the first edition of his introduction to copulas [Nelsen, 1999], reprinted with some new results [Nelsen, 2006].

But, the main reason for this increased interest is to be found in the discovery of the notion of copulas by researchers in several applied fields, like finance. Here we should like briefly to describe this explosion by quoting Embrechts's comments [Embrechts, 2009]:

> As we have seen so far, the notion of copula is both natural as well as easy for looking at multivariate d.f.'s. But why do we witness such an incredible growth in papers published starting the end of the nineties (recall, the concept goes back to the fifties and even earlier, but not under that name). Here I can give three reasons: finance, finance, finance. In the eighties and nineties we experienced an explosive development of quantitative risk management methodology within finance and insurance, a lot of which was driven by either new regulatory guidelines or the development of new products; see for instance Chapter 1 in [McNeil et al., 2005] for the full story on the former. Two papers more than any others "put the fire to the fuse"': the [...] 1998 RiskLab report by Embrechts et al. [2002] and at around the same time, the Li credit portfolio model [Li, 2001].

The advent of copulas in finance, which is well documented in [Genest et al., 2009a], originated a wealth of different investigations: see, for example, the books by Schönbucker [2003], Cherubini et al. [2004], McNeil et al. [2005], Malevergne and Sornette [2006], Trivedi and Zimmer [2007] and the more recent contributions by Jaworski et al. [2010, 2013], Cherubini et al. [2011a] and Mai and Scherer [2012b, 2014]. At the same time, different fields like environmental sciences [Genest and Favre, 2007; Salvadori et al., 2007], biostatistics [Hougaard, 2000; Song, 2007], decision science [Clemen and Reilly, 1999], machine learning [Elidan, 2013], etc., discovered the importance of this concept for constructing more flexible multivariate models. Copulas are used in several commercial statistical software for handling multivariate data and various copula-based packages are included in the R environment (`http://www.r-project.org/`). Nowadays, it is near to impossible to give a complete account of all the applications of copulas to the many fields where they have been used. As Schweizer wrote [Schweizer, 2007]:

> The "era of i.i.d." is over: and when dependence is taken seriously, copulas naturally come into play. It remains for the statistical community at large to recognize this fact.

> And when every statistics text contains a section or chapter on copulas, the subject will have come of age.

However, a word of caution is in order here. Several criticisms have been recently raised about copulas and their applications, and several people started to speak about "copula craze" [Embrechts, 2009].

In the academic literature, a very interesting discussion was raised by Mikosch [2006a,b] (see also the comments by Embrechts [2006], Genest and Rémillard [2006], de Haan [2006], Joe [2006], Lindner [2006], Peng [2006] and Segers [2006]). Starting with a critical point of view to the copula approach that, in his words,

> [...] it promises to solve all problems of stochastic dependence but it falls short in achieving the goal,

Mikosch posed a series of questions that have stimulated in the last years several theoretical investigations about copula models, especially related to statistical inference (see, e.g., [Genest et al., 2009b; Omelka et al., 2009; Choroś et al., 2010; Fermanian, 2013; Bücher et al., 2014] and references therein) and stochastic processes (see, e.g., Kallsen and Tankov [2006]; Bielecki et al. [2010]; Fermanian and Wegkamp [2012]; Patton [2012, 2013]; Härdle et al. [2015]). Nowadays, a decade after Mikosch's comments, we may say that most of his questions have been taken seriously into account and have promoted influential theoretical and practical contributions to the field that "will also remain for the future".

Outside the academic world, instead, the discussion was largely influenced by Salmon [2009] (reprinted in [Salmon, 2012]), which was also considered by Jones [2009]. In the aftermath of the Global financial crisis of 2008–2009, Salmon reported several fallacies of the Li's model for credit portfolio [Li, 2001], corresponding to a Gaussian copula model, and called it "the formula that killed Wall Street", an expression that became suddenly popular. Here the main concern is about the use of Gaussian copulas to describe dependence of default times. As noted by Nassim Nicholas Taleb (as cited in [Salmon, 2009]),

> People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked. Co-association between securities is not measurable using correlation.

Now, it should be stressed that many academics have pointed out the limitations of the mathematical tools used in the finance industry, including Li's formula (see, e.g., [Embrechts et al., 2002; Brigo et al., 2010]). In fact, as summarised by Donnelly and Embrechts [2010],

> One of the main disadvantages of the model is that it does not adequately model the occurrence of defaults in the underlying portfolio of corporate bonds. In times of crisis, corporate defaults occur in clusters, so that if one company defaults then it is likely that other companies also default within a short time period. Under the Gaussian copula model, company defaults become independent as their size of default increases.

However early, these warnings seem to have been often ignored. Li himself understood the fallacy of his model saying "Very few people understand the essence of the model" and "it's not the perfect model" [Whitehouse, 2005]. In our opinion, as often the case in applications, the problem was not about the model *per se*, but its abuse. For a sociological study about model culture and the Gaussian copula story, we refer to MacKenzie and Spears [2014b,a].

However, although both previous criticisms come from different (somehow opposite) perspectives, they were a quite natural reaction to a wide diffusion of theory and applications of copulas, but not always in a well motivated way. It seems that several people have wrongly interpreted copulas as the solution to "all problems of stochastic dependence". This is definitely not the case! Copulas are an indispensable tool for understanding several problems about stochastic dependence, but they cannot be considered the "panacea" for all stochastic models.

Why another book on copulas?

Given the past and recent literature about copulas, made by influential contributions by esteemed colleagues, we should like to justify our decision to write this book.

This book is an attempt to introduce the reader to the present state of the art on copulas. Our starting point is that "*There is nothing as practical as a good theory*". Therefore, further attempts to use copulas in a variety of different applications will remain valid for years (and will not be just fashions) if they are grounded on a solid and formal mathematical background. In this respect, this book should be considered as an *Introduction to Copula Theory*, since it has the ambition to embed the concept of copula into other mathematical fields like probability, real analysis, measure theory, algebraic structures, etc. Whether we have succeeded (at least partially) in our goal is left to the reader's judgment.

In the book, we have tried

- to present the theory of general d-dimensional, $d \geq 2$, copulas;
- to unify various methods scattered through the literature in some common frameworks (e.g., shuffles of copulas);
- to make a deliberate effort to be as simple as possible in proving the results we present, but we are aware that a few proofs are not *proofs from THE BOOK*, according to Erdős's definition, see [Aigner and Ziegler, 1999];
- to find connexions with related functions (quasi-copulas, semi-copulas, triangular norms) that have been used in different domains;
- to give at least an idea of the importance of copulas in applied fields.

In writing this book we have heavily relied on the works of the authors who have written on copulas before us. To this end, two environments have been used in the text, namely, "Historical remarks" and "Further readings", in order to acknowledge the works done by several estimeed scientists during the past years. As the field of copulas is a very active one, the number of new results, of streamlined presentations, of new proofs of old results and the like is very high and the relevant literature is quite large. Therefore, we wish to offer our apologies to those authors who may feel slighted by our overlooking their contribution; we can only excuse ourselves by saying that the literature on copulas has grown to such an extent that it is impossible to quote every paper devoted to some aspect of their theory.

Most of the figures presented in this book were realised in R [R Core Team, 2014] by using the contributed package copula [Hofert et al., 2014]. The authors and maintainers of this software are gratefully acknowledged.

Acknowledgements

We owe a great debt of gratitude to many persons with whom we have had the good fortune to discuss, and from whom we have learned, new aspects of copulas. Many of these researchers have become friends and, in some cases, coauthors. To all of them we address our sincere thanks. In particular, we should like to acknowledge those colleagues and friends that have provided us several interesting suggestions and corrections on preliminary versions of this book.

We express our intellectual and personal indebtedness and the gratitude we feel toward Abe Sklar, the "inventor" of the very concept, and of the name of *copula*, and toward Berthold Schweizer, who together with him, has shown the mathematical world how useful copulas could be. Bert Schweizer did not live to see even the beginning of the writing of this book; we feel much poorer for not having been able to rely, as in so many instances in the past, on the advice he so generously gave us. To them we wish to dedicate this book, whose idea came to us during the meeting held in Lecce in 2009 in order to celebrate the first 50 years of copulas.

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List of symbols

$\mathbb{N}, \mathbb{R}, \overline{\mathbb{R}}$	number sets	Section 1.1
$\mathbb{I}$	closed unit interval	Section 1.1
$\vee, \wedge$	join and meet of a poset	Section 1.1
$\mathbf{u}_j(t)$	$(u_1, \dots, u_{j-1}, t, u_{j+1}, \dots, u_d) \in [0,1]^d$	Section 1.1
$\mathrm{Dom}(F)$	domain of a function F	Section 1.1
$\mathrm{Ran}(F)$	range of a function F	Section 1.1
$\mathbf{1}_A$	indicator function	Section 1.1
r.v.	random vector (of dimension ≥ 1)	Section 1.2
d.f.	(probability) distribution function	Section 1.2
$(\Omega, \mathscr{F}, \mathbb{P})$	probability space	Section 1.2
$\mathscr{B}(A)$	σ-field of Borel sets on A	Section 1.2
λ	1-dimensional Lebesgue measure	Section 1.2
λ_d	d-dimensional Lebesgue measure ($d \geq 2$)	Section 1.2
$\mathscr{D}$	space of (univariate) distribution functions	Section 1.2
$\mathscr{D}^d$	space of d-dimensional distribution functions	Section 1.2
$\mathbf{X} \stackrel{d}{=} \mathbf{Y}$	equality in distribution	Section 1.2
$F^{(-1)}$	quasi-inverse of a univariate distribution function F	Section 1.2
$V_H(B)$	H-volume of a box B	Section 1.2
$\mathscr{C}_d$	the space of d-dimensional copulas	Section 1.3
M_d, Π_d	comonotonicity and independence copula	Section 1.3
W_2	countermonotonicity copula	Section 1.3
$C_\alpha^{\mathbf{CA}}$	(bivariate) Cuadras–Augé copula	Section 1.3
$\lVert\mathbf{u}\rVert_p$	$\ell(p)$ norm	Section 1.3
$C_\alpha^{\mathbf{EFGM}}$	EFGM copula	Section 1.6.1
W_d	Fréchet lower bound	Section 1.7.1
$\Xi(\mathbb{I}^d)$	space of continuous functions on $\mathbb{I}^d$	Section 1.7.2
d_∞	sup-norm in $\Xi(\mathbb{I}^d)$	Section 1.7.2
$\mathrm{Sym}(\mathbb{I}^d)$	group of symmetries of $\mathbb{I}^d$	Section 1.7.3
C^ξ	copula induced from C via $\xi \in \mathrm{Sym}(\mathbb{I}^d)$	Section 1.7.3
C^T	transpose of $C(u,v)$	Section 1.7.3
$\widehat{C}$	survival copula associated with C	Section 1.7.3
$\overline{C}$	survival function associated with C	Section 1.7.3
$C^{\mathbf{FN}}$	Fredricks–Nelsen copula	Section 1.7.3
$\rho(X,Y)$	Spearman's rho of (X,Y)	Section 2.4
$\tau(X,Y)$	Kendall's tau of (X,Y)	Section 2.4

δ, δ_C	diagonal and diagonal section of a copula C	Section 2.6
$\mathscr{C}_{d,\delta}$	subclass of all copulas with a fixed diagonal δ	Section 2.6
$C^{\mathbf{Ber}}$	Bertino copula	Section 2.6
λ_U, λ_L	tail dependence coefficients	Section 2.6.1
μ_C	measure induced by a copula C	Section 3.1
$\mathscr{P}(\mathbb{I}^d)$	set of probability measures on $(\mathbb{I}^d, \mathscr{B}(\mathbb{I}^d))$	Section 3.1
$\mathscr{P}_C(\mathbb{I}^d)$	set of d-fold stochastic measures on $(\mathbb{I}^d, \mathscr{B}(\mathbb{I}^d))$	Section 3.1
$\mu \circ \varphi^{-1}$, $\varphi_{\#}\mu$	image measure (push-forward) of μ under φ	Section 3.1
$\mathscr{M}_d$	the set of transformation arrays	Section 3.3
K_A	Markov kernel related to a copula A	Section 3.4
$\mathscr{T}$	set of measure-preserving transformations of $(\mathbb{I}, \mathscr{B}(\mathscr{I}), \lambda)$	Section 3.6
$\mathscr{T}_p$	set of bijective measure-preserving transformations of $(\mathbb{I}, \mathscr{B}(\mathscr{I}), \lambda)$	Section 3.6
κ_C	Kendall d.f. associated with a copula C	Section 3.9
B_n^C	Bernstein approximation of a copula C	Section 4.1.2
D_∞, D_∂	distances on $\mathscr{C}_d$	Section 4.3
$A * B$	Markov product of the copulas A and B	Section 5.1
$C^{\mathbf{Fre}}$	Fréchet copula	Section 6.2
$\mathscr{G}_d$	class of generators Archimedean copulas	Section 6.5
$C_\alpha^{\mathbf{GH}}$	Gumbel–Hougaard copula	Section 6.5
$C_\alpha^{\mathbf{MTC}}$	Mardia–Takahasi–Clayton copula	Section 6.5
$C_\alpha^{\mathbf{Frank}}$	Frank copula	Section 6.5
$C^{\mathbf{AMH}}$	AMH copula	Section 6.5
$E_d(\mu, \Sigma, g)$	class of elliptical distributions	Section 6.7
$N_d(\mu, \Sigma)$	class of Gaussian distributions	Section 6.7
$C^{\mathbf{Ga}}$	Gaussian copula	Section 6.7
$C^{\mathbf{t}}$	Student's t-copula	Section 6.7
$\mathscr{C}_{d+1}^{\mathbf{LT}}$	set of truncation-invariant copulas	Section 6.8
$\mathscr{C}_2^{\mathbf{LTI}}$	set of irreducible truncation-invariant copulas	Section 6.8
$\mathscr{Q}_d$	set of d-quasi-copulas	Section 7.1
$\mathscr{S}_d$	set of d-semi-copulas	Section 8.1
NBU	"New Better than Used" property	Section 8.6
IFR	"Increasing Failure Rate" property	Section 8.6
$\mathscr{A}_1^+$, $\mathscr{A}_2^+$, $\mathscr{A}_3^+$	classes of multivariate ageing functions	Section 8.6

Contents

1 Copulas: basic definitions and properties **1**
- 1.1 Notations 1
- 1.2 Preliminaries on random variables and distribution functions 2
- 1.3 Definition and first examples 10
- 1.4 Characterisation in terms of properties of d.f.'s 14
- 1.5 Continuity and absolutely continuity 18
- 1.6 The derivatives of a copula 21
  - 1.6.1 Characterisation of bivariate copulas via Dini derivatives 22
- 1.7 The space of copulas 26
  - 1.7.1 Lattice properties 26
  - 1.7.2 Metric properties 28
  - 1.7.3 Symmetry 30
  - 1.7.4 Asymmetrisation of copulas 34
- 1.8 Graphical representations 35

2 Copulas and stochastic dependence **39**
- 2.1 Construction of multivariate stochastic models via copulas 39
- 2.2 Sklar's theorem 42
  - 2.2.1 Sklar's theorem for survival functions 47
- 2.3 Proofs of Sklar's theorem 48
  - 2.3.1 Proof of Sklar's theorem by extension 49
  - 2.3.2 Proof of Sklar's theorem by distributional transform 53
  - 2.3.3 Proof of Sklar's theorem by regularisation 54
- 2.4 Copulas and rank-invariant property 57
- 2.5 Characterisation of basic dependence structures via copulas 63
- 2.6 Copulas and order statistics 68
  - 2.6.1 Diagonal sections of copulas and tail dependence 73

3 Copulas and measures **75**
- 3.1 Copulas and d-fold stochastic measures 75
  - 3.1.1 Doubly stochastic matrices and discrete copulas 78
- 3.2 Absolutely continuous and singular copulas 82
- 3.3 Copulas with fractal support 86
- 3.4 Copulas, conditional expectation and Markov kernel 90
- 3.5 Copulas and measure-preserving transformations 95
- 3.6 Shuffles of a copula 98

3.6.1 Shuffles of Min 101
3.7 Sparse copulas 107
3.8 Ordinal sums 112
3.9 The Kendall distribution function 115

4 Copulas and approximation 123
4.1 Uniform approximations of copulas 123
4.1.1 Checkerboard copulas 123
4.1.2 Bernstein copulas 128
4.1.3 Sparse copulas 131
4.2 Application to weak convergence of multivariate d.f.'s 134
4.3 Markov kernel representation and related distances 137
4.4 Copulas and Markov operators 146
4.4.1 Bivariate copulas and Markov operators 152
4.5 Convergence in the sense of Markov operators 156

5 The Markov product of copulas 161
5.1 The Markov product 161
5.2 Invertible and extremal elements in $\mathscr{C}_2$ 164
5.3 Idempotent copulas, Markov operators and conditional expectations 170
5.4 The Markov product and Markov processes 175
5.5 A generalisation of the Markov product 183

6 A compendium of families of copulas 189
6.1 What is a family of copulas? 190
6.2 Fréchet copulas 192
6.3 EFGM copulas 192
6.4 Marshall-Olkin copulas 194
6.5 Archimedean copulas 196
6.6 Extreme-value copulas 203
6.7 Elliptical copulas 208
6.8 Invariant copulas under truncation 210

7 Generalisations of copulas: quasi-copulas 215
7.1 Definition and first properties 215
7.2 Characterisations of quasi-copulas 218
7.2.1 Proof of Theorem 7.2.4 222
7.3 The space of quasi-copulas and its lattice structure 230
7.3.1 Bounds for copulas with given diagonal 237
7.4 Mass distribution associated with a quasi-copula 239

8 Generalisations of copulas: semi-copulas 249
8.1 Definition and basic properties 249
8.2 Bivariate semi-copulas, triangular norms and fuzzy logic 251
8.3 Relationships between capacities and semi-copulas 255
8.4 Transforms of semi-copulas 259

8.4.1 Transforms of bivariate semi-copulas 267
8.5 Semi-copulas and level curves 271
8.6 Multivariate ageing notions of NBU and IFR 274

Bibliography **285**

Index **313**