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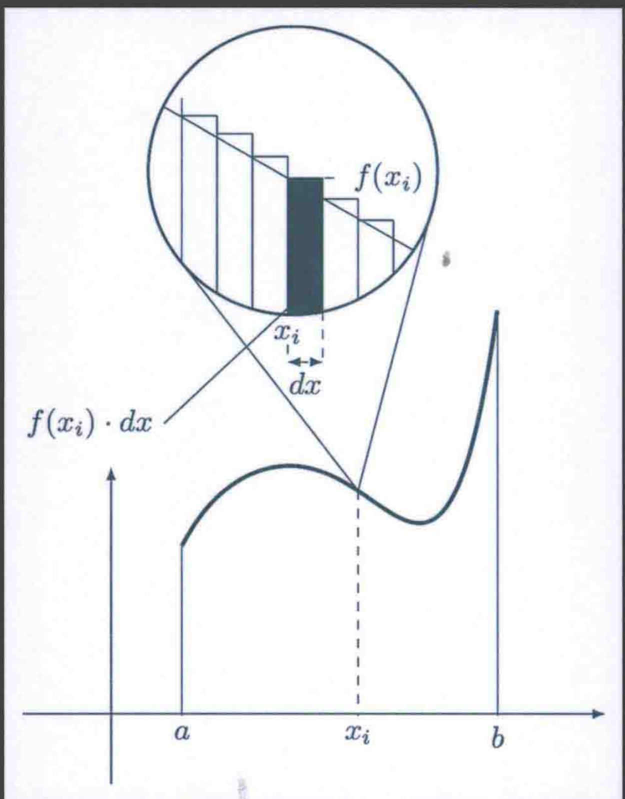
ANALYSIS WITH ULTRASMALL UMBERS



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Preface

Infinitesimals then and now

The first calculus textbook, *Analyse des Infiniment Petits* by the Marquis de L'Hôpital, was published in 1696. As the title indicates, the presentation was based on "infinitely small" or "infinitesimal" quantities, introduced by Gottfried Wilhelm von Leibniz, one of the co-discoverers of calculus. For one hundred and fifty years Leibniz's method of infinitesimals served as the standard way of doing calculus, in preference to Isaac Newton's method of fluxes. It reached high sophistication in the hands of masters such as the Bernoulli brothers and Leonhard Euler. From its inception it was also criticized for the lack of firm foundations (as was Newton's method). Bishop Berkeley [2] famously pointed out the logical discrepancies that appear when dividing by nonzero quantities on the one hand, but then ignoring them in the results as though they were "ghosts of departed quantities" on the other hand.

The work of nineteenth century mathematicians, in particular of Augustin-Louis Cauchy and Karl Weierstrass, succeeded in giving a rigorous treatment of Newton's approach, culminating in the concept of limit defined by the now classical epsilon–delta method. As a result, infinitesimals disappeared from modern mathematical texts. The rigorous foundations provided by the epsilon–delta method enabled an unprecedented flowering of mathematical analysis. Nevertheless, physical scientists have been reluctant to give up on the simplicity and intuitive appeal of infinitesimals, which still persist in some form in contemporary scientific thinking.

A rigorous theory of infinitesimals consistent with the contemporary understanding of mathematical analysis was established in 1960 by Abraham Robinson. His book *Nonstandard Analysis* [24] provided paraphrases of many classical arguments, as well as numerous new results. At the research level, Robinson's methods have found significant applications in analysis, number theory, mathematical physics and other areas of pure and applied mathematics. The underlying framework of nonstandard analysis is model–theoretic, usually based on ultraproducts or superstructures, concepts unsuitable for elementary level exposition; see Goldblatt [4] or Vakil [28] for an excellent graduate level introduction to nonstandard analysis. Even at the research level, the need to invoke model theory is a bothersome distraction from the essential ideas.

In mathematical education the abandonment of infinitesimals had perhaps the greatest impact. The epsilon–delta definition of limit and the proofs based on this definition are just too complicated for an average student to master quickly, if ever. As a result, rigor has disappeared from many modern introductory calculus courses. They are usually taught in a way that leaves the basic concepts undefined and the fundamental theorems unproved. This "faith-based" approach runs counter to the conception of mathematics as a rigorous deductive science that one tries to convey to students in high school algebra and geometry. Many teachers (and some students) are justifiably bothered by this state of affairs.

Some attempts to teach elementary calculus using nonstandard analysis have been made; two nice calculus textbooks in this vein are Keisler [16] and Stroyan [27]. The model–theoretic prerequisites have been circumvented by an axiomatic treatment of an extension of the real number field called the *hyperreals*. Yet it seems fair to say that these attempts have not been as successful as the intuitive simplicity of the concept of infinitesimal would lead one to expect. The third author tried to teach elementary calculus using Keisler's book [16]; this experience and the pedagogical difficulties it uncovered are described in [17]. Besides the need to learn a new non-Archimedean number system while students still struggle to adequately understand the real numbers, there is the need to distinguish between internal and external objects and the potential of the latter to provide distracting, pathological examples. There is also the fact that the infinitesimal definitions of the basic concepts of calculus (derivative, limit, integral) apply only to standard objects. The epsilon–delta definition is still needed to make sense of, say, $f'(x)$ when either x or f is not standard.

Axiomatic nonstandard set theories have been proposed as a way to make nonstandard methods more accessible. Such theories were introduced in the mid-1970s independently by the first author [7, 8], Edward Nelson [18] and Petr Vopěnka [29]; we refer the interested reader to Kanovei and Reeken's comprehensive monograph [14]. Nelson's theory **IST** has found a significant following; see Robert [23] for a nice exposition. The axiomatic framework alleviates some of the pedagogical difficulties of the model–theoretic approach. In the simpler theories, like **IST** or its bounded variant **BST**, there are no external objects and no hyperreals. However, all these axiomatic approaches still have a significant "overhead" of logical formalism. Also, the fixed division of mathematical entities into "standard" and "internal" postulated by these theories means that the last difficulty mentioned above, to wit, that the infinitesimal definitions of the calculus concepts apply only to standard objects, remains in full force (see [19] and [10] for a fuller discussion of this point).

Following an idea of Guy Wallet, Yves Péraire in a series of papers beginning in 1989 ([21] is the most fundamental) developed an axiomatic nonstandard set theory **RIST**, where the notion of "standard" (and, consequently, also of "infinitesimal") is *relative*; every mathematical entity can be regarded as "standard" when viewed in the context of its own appropriate universe. The first author in [9] and [11] strengthened the axioms of Péraire's theory (axiomatic set theories **FRIST** and **GRIST**) and simplified its formalism.

About this book

The theory on which this book is based is a fragment of the bounded version of **RIST** (**RBST**; see the Appendix). It is a result of a long series of simplifications and modifications influenced by classroom experience over a period of ten years. Since the word "standard" in common usage, and even in nonstandard analysis, has an absolute connotation: "usual, ordinary, traditional, prevailing," we use "*observable*" for the relativized version of the concept. Every mathematical object can be regarded as observable relative to a suitable context. The fundamental Principle of Stability asserts, roughly speaking, that objects have exactly the same properties relative to any context where they are observable. In particular, relative to any context there are infinitesimal and infinitely large real numbers (we call them *ultrasmall* and *ultralarge* numbers, respectively, for reasons explained in the Introduction).

A major advantage of the relative approach is that the infinitesimal definitions (of derivative, limit and so on) apply uniformly to all functions and all of their arguments; thus there is no need for the epsilon–delta mechanism. One can completely eliminate it from elementary calculus if one so desires.

An important feature of our approach is the *contextual notation*: notions that depend on the context, such as "observable," "ultrasmall" and "ultralarge," are understood to be relative to the context of the theorem, definition or proof in which they are mentioned (unless explicitly stated otherwise). In conjunction with the Stability Principle, this convention minimizes the need to pay explicit attention to the context and greatly simplifies the presentation. The presentation is axiomatic, based on six principles. The Existence Principle and the Relative Observability Principle set up the basic structure of observability. The Closure Principle asserts, in effect, that objects definable from observable parameters are observable, and the Observable Neighbor Principle asserts that every real number that is not ultralarge has to be ultraclose to some observable real number. The last two important principles, Stability and Definition, are rarely appealed to explicitly; they provide the background justification for the contextual convention.

Of course, we do not expect students (or even trained mathematicians) to prove theorems formally from the axioms. Some intuitive representation of what the axioms are about is necessary. There are in fact two ways to view the axioms (of any nonstandard set theory) intuitively. In the *internal view*, advocated by Nelson, the numbers and sets of the theory are regarded as the usual sets and numbers we are all familiar with. In this view, no new objects are added to the usual mathematical universe; it is only the language that is being extended. The standardness (or, observability) predicate is a linguistic device that singles out some of the familiar objects for special attention. This idea is attractive to those who can reconcile their view of natural numbers with the existence of properties that do not satisfy the Principle of Mathematical Induction. Such properties can be expressed in the extended language; for example, "x is standard" is such a property: 1 is standard; if n is standard, then $n+1$ is standard, but not all natural numbers are standard. We had originally presented the material in this book from the internal point of view (see [13]). It works quite well in the classroom, but it seems that most mathematicians find it incompatible with their ideas about natural numbers.

The alternative is the "standard view" proposed in [7]. In this view, which is adopted in this book, we identify the *standard* (observable in every context) sets with the familiar sets of traditional mathematics. But these sets are seen as also having a plethora of nonstandard, ideal elements, such as the infinitesimal and infinitely large elements of the set $\mathbb{R}$. See Section 1.1 for more details.

Admittedly, this picture still represents a change from the traditional view in which there are no infinitesimals in $\mathbb{R}$, but we think that it should be more easily acceptable. An important point is that the two views differ only philosophically. They are concerned with the intuitive interpretation; the actual mathematics is the same in either view.

The book develops the usual topics from calculus of one real variable. The presentation is based on ultrasmall numbers. It demonstrates that mathematics with ultrasmall numbers can be practiced in a style that is just as informal and natural as the traditional treatments, but with important advantages. Use of ultrasmall numbers is more intuitive and it disposes with the epsilon–delta machinery and with the associated bookkeeping. The proofs become simpler and more focused on the "combinatorial" heart of arguments. Fundamental results, such as the Extreme Value Theorem, can be fully proved from the axioms immediately, without the need to master notions of supremum or compactness. As a result, calculus can be presented as *mathematics*—with proofs—even at a student level where vague arguments about "approaching" have become the norm. Derivatives and definite integrals can be developed before limits, and independently of each other. The relative framework allows arguments involving two or more levels of observability simultaneously. (This is a feature not easily available in the Robinsonian or Nelsonian framework. It simplifies many proofs, especially where double limits are involved.) A rigorous theory of ultrasmall and ultralarge numbers also enables the construction of entirely new models of mathematical and physical phenomena.

Intended audience

In this book, perhaps for the first time, definitions and arguments involving infinitesimals are presented in a style that is both as informal and as rigorous as is customary in standard textbooks of introductory analysis. We eschew both the ultraproduct construction of the model–theoretic nonstandard analysis and the excessive formalism of the axiomatic approaches. This should make the book of interest to a wide audience of mathematically minded readers—mathematicians, teachers of mathematics at high school or college level, scientists and philosophers of mathematics—anybody looking for a simple but rigorous introduction to infinitesimal methods. Although some preliminary acquaintance with calculus would be helpful, the actual prerequisites do not go beyond high school algebra, geometry and trigonometry, making the book, especially Part I, accessible as an independent reading to ambitious beginning calculus students.

This is also the first time that an exposition of the relative framework for nonstandard analysis (allowing many levels of standardness) is given in a book format; until now, it has been available only in research papers. Thus perhaps even experts on nonstandard analysis will find here something of interest.

Our hope for the most significant impact of the book is in the teaching of introductory calculus at the high school or college level. We started this project in response to the high school syllabus of the canton of Geneva (Switzerland), where two of us teach, and which requires courses in calculus (as well as other mathematical subjects) to be taught in the standard mathematical fashion: definition, example, theorem and proof with a reasonable degree of rigor. This turned out to be impossible to do with the traditional epsilon–delta method. Our approach was developed explicitly to satisfy this requirement. It has been used in two Geneva high schools for the last ten years by up to as many teachers, and repeatedly and extensively modified in response to the classroom experience. It has been successful in remedying the situation: It provides simpler definitions for the basic concepts, allowing students to form a good intuition and actually prove things by themselves. Moreover, this approach does not require any additional "black boxes" once the initial axioms have been presented. Many theorems can be proved simply, without resorting to difficult concepts like compactness or completeness. The track record of former students is very encouraging. Those of our students who had to take a course in analysis during their first year at the university all passed the exam. They report no particular difficulties with switching to the standard epsilon–delta method at the university level, having had to work rigorously in analysis before. This contrasts with students exposed to the informal standard method, who encounter rigor in analysis for the first time at the university level. A report on an earlier stage of this project has been published in [20].

For teachers of mathematics who wish to present calculus at an introductory college level, or even high school, with at least some proofs, the text can serve as a reference and a sourcebook of ideas for such a course. This should be of particular interest in countries where proofs are part of the syllabus from the onset, such as Switzerland, France and others. At the introductory level one would aim to cover only some of the material in Part I. In particular, the technical aspects of the Closure and Stability Principles in Chapter 1 can be de-emphasized and/or introduced gradually, as needed in the subsequent chapters. A student handout that illustrates how the ideas from the book can be used at an elementary level is available on the website www.ultrasmall.org.

The format of our book differs from textbooks for traditional Calc 101 courses mainly in that we clearly have to start by convincing the teachers of such courses that ours is a worthwhile approach. They first have to master the techniques themselves, and for this purpose we wrote the book at a slightly higher level, including explanations and material beyond what would be presented to the beginning students. The book is intended to inspire teachers to supplement the usual Calc 101 and 102 material or to fashion their own courses on its basis.

The book is structured so that it could be used as a textbook for a course at a more advanced level, comparable to the (U.S.) first advanced calculus course. In this case, one would probably want to cover most of Parts I and II. This would be especially appropriate for courses directed towards physics or engineering majors, as arguments involving infinitesimals are common in the practice of those fields. We think that there are advantages to teaching with ultrasmall numbers even in a course oriented towards mathematics majors. It seems that many students, even at this level, find it difficult to understand, say, the distinction between pointwise and uniform convergence of a sequence of functions, based on the epsilon–delta definitions of these concepts; an initial approach via infinitesimals might be more intuitive. We recognize that students in a course of this nature have to learn the traditional epsilon–delta methods, and this book makes it possible to get used to them gradually, while maintaining full rigor from the start. The transition to traditional methods is motivated in Section 4.7 (on numerical integration), Chapter 10 (topology of the real line), and explicitly worked out in Section 5.2. We focus on those topics that best illustrate the variety of infinitesimal methods and de-emphasize those where algebraic or computational aspects predominate. (Yet, for the sake of providing a complete course, we also include some theorems whose proofs are not specific to our approach, some routine computational examples and many exercises.) The book could also serve as a text for a seminar or independent study with an emphasis on nonstandard methods.

There are 80 numbered exercises scattered throughout the text. They are an important part of the learning experience and the reader is encouraged to attempt all of them. In many cases, the results are used later in the text. They all have worked out solutions starting on page 241. Additional exercises (without answers) are placed at the end of each chapter (170 in all), ranging from the routine to the more challenging.

Chapter-by-chapter summary

Part I includes material that—probably with omission of some of the more difficult proofs—could be covered in an elementary calculus course. In an advanced calculus course one would want to include all the proofs.

Chapter 1 provides some intuition about how to interpret the nontraditional concept of observability on which our approach is based. It formulates the basic principles that govern observability and defines the key concepts: ultrasmall and ultralarge numbers and observable neighbors.

Chapter 2 studies continuity and limits. In particular, simple proofs of the Intermediate Value Theorem and the Extreme Value Theorem are given; they do not rely on the notion of supremum or topological properties such as compactness. Uniform continuity is also introduced, and the theory of exponentiation with real exponents is developed.

Chapter 3 develops elementary differential calculus and Chapter 4, integration of continuous functions. All relevant theorems are fully proved.

Part II contains material that would not usually be found in a first calculus course, but that should be included in advanced calculus.

Sections 5.1 and 5.2 in Chapter 5 discuss the notion of supremum, completeness of the real numbers, mathematical induction, and the epsilon–delta method. With the exception of induction, this material is almost never used in the rest of the book and can be omitted or postponed. Section 5.3 establishes a useful equivalent version of the definition of limit.

Chapter 6 proves various versions of L'Hôpital's Rule, introduces higher derivatives, and defines the Taylor polynomial.

Chapter 7 develops the usual material on sequences and series in our framework. Uniform convergence of sequences of functions is studied in Section 7.4.

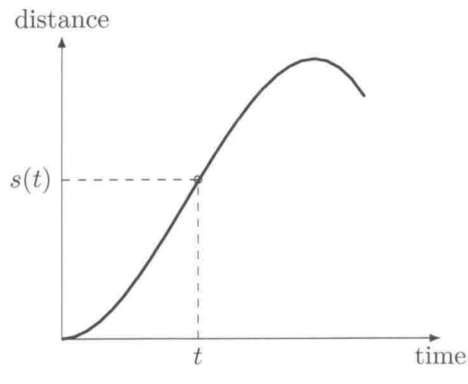
The last three chapters of Part II are independent of each other. Chapter 8 begins with some elementary material on differential equations, and then follows with a nonstandard proof of the Peano theorem about the existence of solutions of first order differential equations. The proof of the uniqueness theorem assuming the Lipschitz condition is also given. Chapter 9 develops the theory of the Riemann integral. Chapter 10 illustrates the nonstandard treatment of topological concepts, such as open, closed, dense and compact sets, in the simple setting of sets of real numbers.

The Appendix, intended for mathematically more sophisticated readers, gives a formal outline of the foundations on which our approach rests. After a brief review of logical notation and the role of axioms and proofs, we state formally the axioms of the nonstandard set theory **RBST** and deduce from them the principles used in the text. We then discuss consistency of **RBST** and its extensions and provide a guide to the history and literature of the subject.

Preface for Students

Calculus was developed by Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716) in the last third of the seventeenth century as a general method for the study of changing quantities (functions). It has found extensive applications in every field of science concerned with change: physics, chemistry, geology, ecology, economics; in engineering, finance and many other areas. Newton and Leibniz discovered calculus independently and approached it from different viewpoints. In order to understand the difference, let us look at a simple example of an important problem of calculus.

We consider a point-like object P moving in a straight line. The position of P at time t is determined by the distance $s(t)$ of P from a fixed origin O.

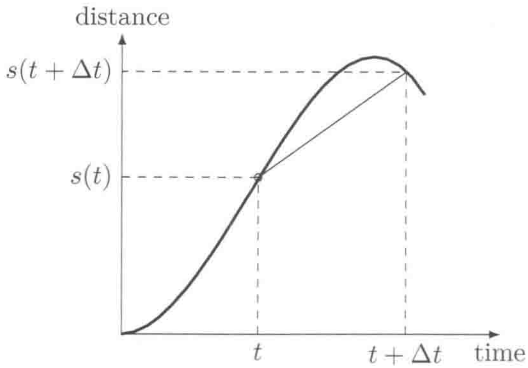


A fundamental assumption of mechanics is that the moving object has, at each time t, a definite *instantaneous velocity* $v(t)$, and one of its basic problems is to determine this instantaneous velocity, assuming that the distance function is known.

We begin by observing that the *average velocity in an interval*, say from t to $t + \Delta t$ where $\Delta t > 0$, can be obtained by a straightforward algebraic computation.

If $s(t)$ is the distance of the object from the origin at time t, $s(t+\Delta t)$ is its distance from the origin at time $t + \Delta t$, hence, during the time interval from t to $t + \Delta t$ the object has travelled the net distance Δs equal to $s(t + \Delta t) - s(t)$, with the average velocity

$$\frac{\Delta s}{\Delta t} = \frac{s(t + \Delta t) - s(t)}{\Delta t}. \tag{1}$$



As an instant has no measurable duration, one might think that the instantaneous velocity $v(t)$ at time t could be obtained from equation (1) by setting $\Delta t = 0$. However, this idea does not work because the resulting expression $0/0$ is mathematically meaningless. It does not follow that there is *no way* of obtaining $v(t)$ from equation (1); however, to do so we have to employ some reasoning, in addition to algebra.

Let us consider a specific example: a small ball in free fall. It has been determined experimentally by Galileo Galilei (1564–1642) that the distance of the falling ball from the point of release is $s(t) = ct^2$, where the constant c has approximate numerical value 5 (if time is measured in seconds and distance in meters). For an object moving according to $s(t) = 5t^2$ we have

$$\Delta s = 5(t + \Delta t)^2 - 5t^2 = 10t(\Delta t) + 5(\Delta t)^2$$

and the average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{10t(\Delta t) + 5(\Delta t)^2}{\Delta t} = 10t + 5\Delta t. \tag{2}$$

Can the instantaneous velocity at time t be obtained from this formula? Intuitively, the instantaneous velocity is approximated by the average velocity when Δt is very small, and it has to depend only on the time t, not on the arbitrary choice of Δt we use to compute the average velocity. The expression on the right side of equation (2) is a sum of two terms: the term $10t$ that depends only on t, and the term $5\Delta t$ that depends on Δt; moreover, if Δt is very small, this second term is also very small. We conclude that the first term $10t$ represents the instantaneous velocity $v(t)$ at time t, and the second term $5\Delta t$ represents the difference between $v(t)$ and the average velocity in the interval $[t, t + \Delta t]$. The challenge is to convert this reasoning into rigorous mathematics.