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Fixed Point Theory under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications

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To
my mother Sania, my father Ali,
my wife Fadoua, my children Adam and Rahma,
my brothers Sofien and Mohamed Amin,
my sister Elhem,
my mother-in-law Zineb, my father-in-law Ridha, and
all members of my extended family ....

Aref Jeribi

To
the memory of my mother Jalila,
my father Hassan,
my wife Nozha, and my children Mohamed and Zaineb.

Bilel Krichen

Preface

This book focuses on fixed point theory for block operator matrices and its applications to a wide range of diverse equations such as, e.g., transport equations arising in the kinetic theory of gas (see [63]), stationary nonlinear biological models (see [138]), and in particular two-dimensional boundary value problems arising in growing cell populations and functional systems of integral equations. In all these topics, we are faced with problems such as the loss of compactness of mappings and/or missing appropriate geometric and topological structure of their underlying domain. Hence, it is convenient that we focus on fixed point results under the weak topology.

As a general rule, emphasis is generally put on some generic and recent results regarding the main topic, since it would be impossible to aim for complete coverage. Simultaneously throughout the chapters (5–7), we tried to illustrate the diversity of the theoretical results in the different settings (Banach spaces and Banach algebras).

In recent years, a number of excellent monographs and surveys presented by distinguished authors about fixed point theory have appeared such as, e.g., [2, 3, 45, 88]. Most of the above mentioned books deal with fixed point theory related to continuous mappings in topology and all its modern extensions. However, it is not always possible to show that a given operator between Banach spaces is weakly continuous. Quite often, its weak sequential continuity does not present any problem. As a first aim, this book is devoted to the study of several extensions of Schauder's and Krasnosel'skii's fixed point theorems to the class of weakly compact operators acting on Banach spaces as well as Banach algebras, in particular on spaces satisfying the Dunford–Pettis property. Notice that both of the above mentioned theorems can be used in order to resolve some open problems, seen in [99] and [115]. We first give some extension forms of Schauder's theorem by using some tools of the weak topology. Then, we present other results which are deduced by quite simple arguments. The notion of weak sequential continuity seems to be the most convenient one to use. It is not always possible to show that a map is weakly continuous. However, weakly sequentially continuous maps are shown to be the most convenient ones to use. That is why some new variants of fixed point theorems involving the measure of weak noncompactness and based on the notion of weak sequential continuity are presented in Banach spaces as well as Banach algebras. Some nonlinear alternatives for the sum of two weakly sequentially continuous mappings, and belonging to Leray–Schauder and Furi–Pera, are also presented.

The second objective of this book is dealing with the following question: Under which conditions will a 2×2 block operator matrix with nonlinear entries (and acting on Banach spaces and Banach algebras) have a fixed point? Based on the previous extension established under the weak topology setting, we are planning to extend these results by proving that, under certain hypotheses associated with its nonlinear entries, the 2×2 block operator matrix

$$\mathcal{L} := \begin{pmatrix} A & B.B' \\ C & D \end{pmatrix}$$

has a fixed point in $\Omega \times \Omega'$, where Ω and Ω' constitute two nonempty, closed, and convex subsets of Banach spaces or Banach algebras. This discussion is based on the presence or absence of the invertibility of the diagonal terms of $\mathcal{I} - \mathcal{L}$. Several fixed point theorems from Chapter 2 enable us to get new results for a particular 2×2 block operator matrix involving operators such as, e.g., $\mathcal{D}$-Lipschitzian, convex-power condensing, weakly sequentially continuous..., acting on a Banach algebra X. A regular case is considered when X is a commutative Banach algebra satisfying the so-called condition $(\mathcal{P})$; that is, for any sequences $\{x_n\}$ and $\{y_n\}$ in a Banach algebra X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then $x_n.y_n \rightharpoonup x.y$; here $\rightharpoonup$ denotes the weak convergence. That condition is very important and plays a key role in the investigations conducted in the proposed monograph. A new recent work is considered, when the entries are assumed multi-valued mappings.

Many applications to a wide range of diverse equations such as, e.g., transport equations arising in the kinetic theory of gas [97], stationary nonlinear biological models, in particular the two-dimensional boundary value problem arising in a growing cell population, a functional system of integral equations, and differential inclusions, are presented.

We should emphasize that this book is the first one dealing with the topological fixed point theory for block operator matrices with nonlinear entries in Banach spaces and Banach algebras. This book can also be regarded as a modest contribution to the fixed point theory in Banach spaces and Banach algebras. Researchers, as well as graduate students in applicable analysis, will find that this book constitutes a useful survey of the fundamental principles of the subject. Nevertheless, the reader is assumed to be, at least, familiar with some related sections concerning notions like the fixed point theorems of Schauder, Krasnosel'skii, Leray–Schauder and Furi–Pera, the basic tools of the weak topology, the concept of measures of weak noncompactness, and the transport equations, etc. Otherwise, the reader is urged to consult the recommended literature in order to benefit fully from this book.

There are some theorems which are based upon a number of hypotheses. These results are very recent. We should notice that, in a future and improved version of this book, the number of hypotheses in some theorems could be lowered and yet lead to the same conclusions. In other words, it may be possible to find an optimum number of hypotheses in the future.

The first author should mention that in the thesis work performed under his direction, by his former students and present colleagues Afif Ben Amar, Ines Feki, Bilel Krichen, Soufien Chouayekh, Rihab Moalla, Bilel Mefteh, Najib Kaddachi, and Wajdi Chaker, the obtained results have helped us in writing this monograph.

Finally, we would like to thank our friends and colleagues, particularly Pr. Ridha Damak whose encouragement and valuable remarks influenced the development of this monograph. We are grateful to Dr. Bilel Mefteh, Dr. Wajdi Chaker, Dr. Rihab Moalla, and Dr. Najib Kaddachi, who have read and commented upon the entire manuscript. Their constructive criticism has led to many improvements and has been a very great help. We encourage any comments or suggestions from any researcher. These comments or suggestions will certainly enable us to integrate some improvements for a new version of this book.

Aref Jeribi and *Bilel Krichen*

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Symbol Description

The most frequently used notations, symbols, and abbreviations are listed below.

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$, $\mathbb{C}$	The fields of real and complex numbers, respectively.
$\mathbb{R}^n$	The n-dimensional real space.
$\inf(A)$	The infimum of the set A.
$\sup(A)$	The supremum of the set A.
∂A	The boundary of the set A
$\partial_\Omega A$	The boundary in Ω of the set A.
B_r	The ball with radius r.
B_X	The unit ball in X.
$d(x, y)$	The distance between x and y.
(X, d)	The metric space X.
X^*, X'	The dual of X.
$diam(A)$	The diameter of the set A, where A is a subset of a metric space X.
$dim(X)$	The dimension of the space X.
$\|x\|$	The norm of x.
$(X, \|.\|)$	The linear normed space X.
$\{x_n\}$	The sequence $\{x_n\}$.
$(\mathcal{P})$	The $(\mathcal{P})$ property.
BOM	Block Operator Matrix.
$\mathcal{P}(X)$	Set of subset of X.
$Gr(T)$	The graph of T.
Φ_T	The $\mathcal{D}$-function of the mapping T.
$\mathcal{P}_{cl}(S)$	The family of all closed subsets of S.
$\mathcal{P}_{cv}(S)$	The family of all convex subsets of S.
$\mathcal{P}_{cl,cv}(S)$	The family of all closed convex subsets of S.
l.s.c	Lower semi-continuous.
u.s.c	Upper semi-continuous.
$\mathcal{C}(\Omega, \mathbb{R})$	The space of real continuous functions on Ω.
$\mathcal{C}_b(I)$	The space of real continuous and bounded functions on I.
$\mathcal{C}(J, X)$	The Banach algebra of real continuous functions from J into X.
$\mathcal{C}(J, \mathbb{R})$	The Banach algebra of real continuous functions from J into $\mathbb{R}$.
$\mathcal{C}(J)$	The Banach algebra of real continuous functions from J into $\mathbb{R}$.
L_p	The L_p space.

$\mathcal{W}(X,Y)$	The set of weakly compact linear operators from X into Y.
$\mathcal{W}(X)$	The ideal of weakly compact linear operators on X.
$\mathcal{L}(X,Y)$	The space of linear operators from X into Y.
$\mathcal{L}(X)$	The space of linear operators from X into X.
$x_n \rightharpoonup x$	x_n converges weakly to x.
$x_n \to x$	x_n converges strongly to x.
$\overline{M}$	The closure of the set M.
$\overline{M^w}$	The weak closure of the set M.
$co(.)$	The convex hull.
$conv(.)$	The convex hull.
$\overline{co}(.)$	The closed convex hull.
$\overline{conv}(.)$	The closed convex hull.
(S,Σ,μ)	The measure space S.
MNWC	Measure of weak noncompactness.
DP	Dunford–Pettis property.
FIE	Functional Integral Equation.
FDE	Functional Differential Equation.
(m)	The (m) property.
T_H	The streaming operator.
$\mathcal{N}_f$	Nemytskii's operator generated by f.
IVP	Initial Value Problem.

Contents