

**INTERNATIONAL SERIES OF MONOGRAPHS IN
PURE AND APPLIED MATHEMATICS**

GENERAL EDITORS: I. S. SNEDDON AND M. STARK

EXECUTIVE EDITORS: J. P. KAHANE, A. P. ROBERTSON AND S. ULAM

VOLUME 101

**INTRODUCTION TO SET
THEORY AND TOPOLOGY**

COMPLETELY REVISED SECOND ENGLISH EDITION

INTRODUCTION TO SET THEORY AND TOPOLOGY

KAZIMIERZ KURATOWSKI

Professor of Mathematics
Member of the Polish Academy of Sciences

Containing a Supplement
on
ELEMENTS OF ALGEBRAIC TOPOLOGY
by

Professor RYSZARD ENGELKING

COMPLETELY REVISED SECOND ENGLISH EDITION
FIRST EDITION TRANSLATED FROM POLISH BY LEO F. BORON

PWN-POLISH SCIENTIFIC PUBLISHERS
WARSZAWA

PERGAMON PRESS
OXFORD • NEW YORK • TORONTO • SYDNEY
PARIS • FRANKFURT

Graphic design: *Zygmunt Ziemka*

This copy is to be sold only in Poland, Albania, Bulgaria, Chinese People's Republic, Czechoslovakia, Cuba, German Democratic Republic, Hungary, Korean People's Democratic Republic, Mongolia, Rumania, Vietnam, U.S.S.R., Yugoslavia, and not for export therefrom.

Copyright © 1972 by PWN—Polish Scientific Publishers—Warszawa

All Rights Reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means: electronic, electrostatic, magnetic tape, mechanical, photocopying, recording or otherwise, without permission in writing from the publishers

First Edition 1962

Second Edition 1972

Reprinted with revisions 1977

Printed in Poland

CONTENTS

Foreword to the first English edition	11
Foreword to the second English edition	13

Part I

Set Theory

Introduction to Part I	17
----------------------------------	----

I. PROPOSITIONAL CALCULUS

§ 1. The disjunction and conjunction of propositions	23
§ 2. Negation	24
§ 3. Implication	25
Exercises	26

II. ALGEBRA OF SETS. FINITE OPERATIONS

§ 1. Operations on sets	27
§ 2. Inter-relationship with the propositional calculus	28
§ 3. Inclusion	30
§ 4. Space. Complement of a set	32
§ 5. The axiomatics of the algebra of sets	33
§ 6. Boolean algebra. Lattices	34
§ 7. Ideals and filters	36
Exercises	36

III. PROPOSITIONAL FUNCTIONS (PREDICATES)

§ 1. The operation $\{x: \varphi(x)\}$	39
§ 2. Quantifiers	40
§ 3. Ordered pairs	42
§ 4. Cartesian product	42
§ 5. Propositional functions of two variables. Relations	43
§ 6. Cartesian products of n sets. Propositional functions of n variables	46
§ 7. On the axiomatics of set theory	47
Exercises	49

IV. THE MAPPING CONCEPT. INFINITE OPERATIONS. FAMILIES OF SETS

§ 1. The mapping concept	50
§ 2. Set-valued mappings	52
§ 3. The mapping $F_x = \{y: \varphi(x, y)\}$	53
§ 4. Images and inverse images determined by a mapping	54
§ 5. The operations $\bigcup \mathcal{R}$ and $\bigcap \mathcal{R}$. Covers	55
§ 6. Additive and multiplicative families of sets.	56
§ 7. Borel families of sets	58
§ 8. Generalized cartesian products	59
Exercises	60

V. THE CONCEPT OF THE POWER OF A SET. COUNTABLE SETS

§ 1. One-to-one mappings	65
§ 2. Power of a set	67
§ 3. Countable sets	68
Exercises	71

VI. OPERATIONS ON CARDINAL NUMBERS. THE NUMBERS \aleph AND \mathfrak{c}

§ 1. Addition and multiplication	73
§ 2. Exponentiation	75
§ 3. Inequalities for cardinal numbers	79
§ 4. Properties of the number \mathfrak{c}	81
Exercises	84

VII. ORDER RELATIONS

§ 1. Definitions	85
§ 2. Similarity. Order types	85
§ 3. Dense ordering	87
§ 4. Continuous ordering	87
§ 5. Inverse systems, inverse limits	88
Exercises	90

VIII. WELL ORDERING

§ 1. Well ordering	91
§ 2. Theorem on transfinite induction	92
§ 3. Theorems on the comparison of ordinal numbers.	92
§ 4. Sets of ordinal numbers	95
§ 5. The number \aleph_1 (denoted also ω_1).	95
§ 6. The arithmetic of ordinal numbers	97

§ 7. The well-ordering theorem	100
§ 8. Definitions by transfinite induction	102
Exercises	105

Part II

Topology

Introduction to Part II	109
-----------------------------------	-----

IX. METRIC SPACES. EUCLIDEAN SPACES

§ 1. Metric spaces	115
§ 2. Diameter of a set. Bounded spaces. Bounded mappings	116
§ 3. The Hilbert cube	117
§ 4. Convergence of a sequence of points	117
§ 5. Properties of the limit	118
§ 6. Limit in the cartesian product	119
§ 7. Uniform convergence	121
Exercises	122

X. TOPOLOGICAL SPACES

§ 1. Definition. Closure axioms	123
§ 2. Relations to metric spaces	123
§ 3. Further algebraic properties of the closure operation	125
§ 4. Closed sets. Open sets	126
§ 5. Operations on closed sets and open sets	127
§ 6. Interior points. Neighbourhoods	129
§ 7. The concept of open set as the primitive term of the notion of topological space	130
§ 8. Base and subbase	131
§ 9. Relativization. Subspaces	132
§ 10. Comparison of topologies	132
§ 11. Cover of a space	133
Exercises	134

XI. BASIC TOPOLOGICAL CONCEPTS

§ 1. Borel sets	137
§ 2. Dense sets and boundary sets	138
§ 3. \mathcal{T}_1 -spaces. \mathcal{T}_2 -spaces	139
§ 4. Regular spaces, normal spaces	139
§ 5. Accumulation points. Isolated points	141
§ 6. The derived set	141
§ 7. Sets dense in themselves	142
Exercises	143

XII. CONTINUOUS MAPPINGS

§ 1. Continuity	145
§ 2. Homeomorphisms	147
§ 3. Case of metric spaces	149
§ 4. Distance of a point from a set. Normality of metric spaces	154
§ 5. Extension of continuous functions. Tietze theorem	156
§ 6. Completely regular spaces	162
Exercises	163

XIII. CARTESIAN PRODUCTS

§ 1. Cartesian product $X \times Y$ of topological spaces	165
§ 2. Projections and continuous mappings	166
§ 3. Invariants of cartesian multiplication	167
§ 4. Diagonal	168
§ 5. Generalized cartesian products	169
§ 6. X^T considered as a topological space. The cube \mathcal{I}^T	170
§ 7. Cartesian products of metric spaces	172
Exercises	173

XIV. SPACES WITH A COUNTABLE BASE

§ 1. General properties	176
§ 2. Separable spaces	177
§ 3. Theorems on cardinality in spaces with countable bases	178
§ 4. Imbedding in the Hilbert cube	179
§ 5. Condensation points. The Cantor-Bendixson theorem	181
Exercises	183

XV. COMPLETE SPACES

§ 1. Complete spaces	185
§ 2. Cantor theorem	186
§ 3. Baire theorem	186
§ 4. Extension of a metric space to a complete space	188
Exercises	189

XVI. COMPACT SPACES

§ 1. Definition	190
§ 2. Fundamental properties of compact spaces	190
§ 3. Cartesian products	192
§ 4. Compactification of completely regular spaces	195
§ 5. Compact metric spaces	197
§ 6. The topology of uniform convergence of Y^X	207

§ 7. The compact-open topology of Y^X	208
§ 8. The Cantor discontinuum	210
§ 9. Continuous mappings of the Cantor discontinuum	213
Exercises	216

XVII. CONNECTED SPACES

§ 1. Definition. Separated sets.	223
§ 2. Properties of connected spaces	225
§ 3. Components.	229
§ 4. Cartesian products of connected spaces	231
§ 5. Continua	232
§ 6. Properties of continua	233
Exercises	237

XVIII. LOCALLY CONNECTED SPACES

§ 1. Definitions and examples	240
§ 2. Properties of locally connected spaces	240
§ 3. Locally connected continua	243
§ 4. Arcs. Arcwise connectedness	245
§ 5. Continuous images of intervals	246
Exercises	252

XIX. THE CONCEPT OF DIMENSION

§ 1. 0-dimensional spaces	254
§ 2. Properties of 0-dimensional metric separable spaces	254
§ 3. n -dimensional spaces	255
§ 4. Properties of n -dimensional metric separable spaces	257
Exercises	258

XX. SIMPLEXES AND THEIR PROPERTIES

§ 1. Simplexes	259
§ 2. Simplicial subdivision	261
§ 3. Dimension of a simplex	265
§ 4. The fixed point theorem	267
Exercises	270

XXI. CUTTINGS OF THE PLANE

§ 1. Auxiliary properties of polygonal arcs	273
§ 2. Cuttings	274
§ 3. Complex functions which vanish nowhere. Existence of the logarithm	275

§ 4. Auxiliary theorems	276
§ 5. Corollaries to the auxiliary theorems	280
§ 6. Theorems on the cuttings of the plane	282
§ 7. Janiszewski theorems	284
§ 8. Jordan theorem	285
Exercises	289

Supplement

Elements of Algebraic Topology
by R. Engelking

Introduction	291
§ 1. Complexes. Polyhedra. Simplicial approximation	292
§ 2. Abelian groups	298
§ 3. Categories and functors	303
§ 4. Homology groups of simplicial complexes	306
§ 5. Chain complexes	316
§ 6. Homology groups of polyhedra	322
§ 7. Homology groups with coefficients	329
§ 8. Cohomology groups	333
Exercises	338
LIST OF IMPORTANT SYMBOLS	343
INDEX	345
OTHER TITLES IN THE SERIES	350

FOREWORD TO THE FIRST ENGLISH EDITION

The ideas and methods of set theory and topology penetrate modern mathematics. It is no wonder then that the elements of these two mathematical disciplines are now an indispensable part of basic mathematical training. Concepts such as the union and intersection of sets, countability, closed set, metric space, and homeomorphic mapping are now classical notions in the whole framework of mathematics.

The purpose of the present volume is to give an accessible presentation of the fundamental concepts of set theory and topology; special emphasis being placed on presenting the material from the viewpoint of its applicability to analysis, geometry, and other branches of mathematics such as probability theory and algebra. Consequently, results important for set theory and topology but not having close connections with other branches of mathematics, are given a minor role or are omitted entirely. Such topics are, for instance, investigations on foundations, the theory of alephs, and the theory of curves.

The main body of the book is an introduction to set theory and topology, intended for the beginner. Sections marked with an asterisk cover either more complicated topics or points which are frequently omitted in a first course; this holds also for some exercises which allow the reader to get acquainted with many applications and some important results which could not be included in the text without unduly expanding it. Many new exercises not contained in the Polish edition have been included here.

I take great pleasure in thanking Professor J. Jaworowski and Dr. A. Granas for their cooperation in preparing the Polish edition and to thank also Professors A. Mostowski and R. Sikorski, Dr. S. Mrówka, Mr. R. Engelking and Dr. A. Schinzel for numerous comments which helped me to improve the original manu-

script. Also, my thanks go to Mr. Leo F. Boroń and to Mr. A.H. Robinson for preparing the present text for English-speaking students of mathematics.

KAZIMIERZ KURATOWSKI

Warsaw
September 1960

FOREWORD TO THE SECOND ENGLISH EDITION

Since the first English edition appeared, set theory and point-set topology have developed to such an extent that the author found it necessary to modify in many points the previous edition. This was done partially in the Polish edition (1965) and in the French edition (1966).

The most essential changes concern the second part of this book (devoted to topology). However, there are also changes worth noticing in the first part (on set theory).

The concepts of inverse limit, of lattice, of ideal, of filter, of a commutative diagram, and of a cartesian product of an arbitrary number of factors are considered. A slightly deeper insight into the axioms of set theory was needed; in particular, the notion of a class (in the sense of Bernays) is mentioned (and later applied, mainly in connection with the concept of category used in the Supplement).

In the theory of ordering relations, more emphasis was put on what was previously called partial ordering. This is now called, more concisely, ordering, and this change of terminology seems to be more appropriate to common use.

For the same reason, some notations have been changed. In particular, the Lebesgue notation $E_x \varphi(x)$ has been replaced by $\{x: \varphi(x)\}$; the union of members of a family A of sets is denoted by $\bigcup A$, and the intersection by $\bigcap A$.

The changes in the second part of the book are more essential. In the first edition, this part of the book was chiefly devoted to the study of metric spaces. In this second edition, the general topological spaces form its main subject. Consequently, more than a half of the second part had to be written anew. It contains new topics which were not considered in the first edition, such as cartesian products of topological spaces, the Čech-Stone com-

pactification, quotient-spaces, completely regular spaces, quasi-components, and a large number of exercises have been added.

In Chapter XX, on simplexes, more material will be found on simplicial mappings, on the nerve of a cover and related problems.

Finally, the rather short Chapter XXI, on complexes, chains and homologies, has been replaced by a much more extensive Supplement on the elements of algebraic topology. This Supplement, written by Professor Engelking, will certainly be a very valuable complement to my book.

I have received considerable help from the persons mentioned in the Foreword to the first edition and also from the young ladies Dr. Karłowicz and Dr. Vuilleumier. To all of them go my heartiest thanks.

KAZIMIERZ KURATOWSKI

Warsaw
October 1968

Part I

SET THEORY

